PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XII.

FROM NOVEMBER, 1880, TO NOVEMBER, 1881.

LONDON:

C. F. HODGSON & SON, GOUGH SQUARE,

FLEET STREET.

LONDON:
PRINTED BY C. F. HODGSON AND SON,
GOUGH SQUARE, FLEET STREET, E.C.

VOLUMES ALREADY ISSUED.

VOL.	I.	(from January 1865 to November 1866)	10*s*.
,,	II.	(from November 1866 to November 1869)	16*s*.
,,	III.	(from November 1869 to November 1871)	20*s*.
,,	IV.	(from November 1871 to November 1873)	31*s*. 6*d*.
,,	V.	(from November 1873 to November 1874)	15*s*.
,,	VI.	(from November 1874 to November 1875)	21*s*.
,,	VII.	(from November 1875 to November 1876)	21*s*.
,,	VIII.	(from November 1876 to November 1877)	21*s*.
,,	IX.	(from November 1877 to November 1878)	21*s*.
,,	X.	(from November 1878 to November 1879)	18*s*.
,,	XI.	(from November 1879 to November 1880)	12*s*. 6*d*.

PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XII.

SEVENTEENTH SESSION, 1880–81.

November 11*th*, 1880.

ANNUAL GENERAL MEETING, held at 22, Albemarle Street.

C. W. MERRIFIELD, Esq., F.R.S., President, in the Chair.

The Treasurer (S. Roberts, Esq., F.R.S.) read his Report, which, on the motion of Mr. H. M. Jeffery, F.R.S., seconded by Lieut.-Col. J. R. Campbell, and supported by Mr. A. B. Kempe (who called attention to the fact of Mr. Roberts's resignation of the office of Treasurer, and congratulated that gentleman on the favourable Report he had laid before the Members), was accepted.

At the request of the Chairman, Mr. J. Stirling consented to act as Auditor.

The Secretaries' Report was read and adopted.

From the Report of the Secretaries, it appeared that the number of ordinary Members during the past Session had increased from 141 to 151, of whom 52 have compounded.

The communications made to the Society during the past Session had been as follows:—

"On the Binomial Equation $x^p-1=0$, Trisection and Quartisection:" Prof. Cayley, F.R.S.

"On Cubic Determinants and other Determinants of Higher Class, and on Determinants of Alternate Numbers:" Mr. R. F. Scott, M.A.

"On a Problem of Fibonacci's:" Mr. S. Roberts, F.R.S.

"Notes on a Class of Definite Integrals:" Mr. T. R. Terry, M.A.

"Note on a Method of obtaining the q-Formula for the Sine-amplitude in Elliptic Functions:" Mr. J. W. L. Glaisher, F.R.S.

"Note on a Numerical Theorem connected with the Cubical Division of Space:" the President.

"Notes on Curvature:" Mr. J. J. Walker, M.A.

"A Property of a Linkage:" Mr. A. B. Kempe, B.A.

"A Theorem in Spherical Trigonómetry:" Prof. Cayley, F.R.S.

"Geometrical Notes (3):" Prof. H. J. S. Smith, F.R.S.

"On the Reflection of Vibrations at the Confines of Two Media between which the Transition is Gradual," and "on the Stability or Instability of certain Fluid Motions:" Lord Rayleigh, F.R.S.

"The Calculus of Equivalent Statements (iv.):" Mr. Hugh McColl, B.A.

"Notes on a General Method of Solving Partial Differential Equations of the First Order with several Dependent Variables:" Mr. H. W. Lloyd Tanner, M.A.

"Note on the Integral Solution of $x^2-2Py^2=-z^2$ or $\pm 2z^2$ in Certain Cases:" Mr. S. Roberts, F.R.S.

"Notes (1) on a Geometrical Form of Landen's Theorem with regard to a Hyperbolic Arc," (2) "On a Class of Closed Ovals whose Arcs possess the same property as two Fagnanian Arcs of an Ellipse:" Mr. J. Griffiths, M.A.

"A Form of the Equations determining the Foci and Directrices of a Conic whose Equation in Cartesian Coordinates is given:" Prof. Wolstenholme, M.A.

"The Application of Elliptic Coordinates and Lagrange's Equations of Motion to Euler's Problem of two Centres of Force:" Prof. Greenhill, M.A.

"Theorems in the Calculus of Operations:" Mr. J. J. Walker, M.A.

"On the Equilibrium of Cords and Beams in Certain Cases:" Mr. W. J. Curran Sharp, M.A.

"On Steady Motion and Vortex Motion in an Incompressible Viscous Fluid:" Mr. T. Craig, M.A.

"On Functions Analogous to Laplace's Functions:" Mr. E. J. Routh, F.R.S.

"On Cremonian Congruences:" Dr. Hirst, F.R.S.

"On some Statical and Kinematical Theorems:" Prof. Minchin, M.A.

"On a Class of Analytical Problems:" Prof. Cayley, F.R.S.

"On a Binomial Biordinal and the Arbitrary Constants of its complete Solution:" Sir James Cockle, F.R.S.

"On the Focal Conics of a Bicircular Quartic:" Mr. Harry Hart, M.A.

"Preliminary Note on a Generalisation of Pfaff's Problem:" Mr. H. W. Lloyd Tanner, M.A.

"On the Resultant of a Cubic and a Quadric Binary Form:" Prof. Cayley, F.R.S.

"On the Theory of the Focal Distances of Points on Plane Curves:" Mr. W. J. Curran Sharp, M.A.

"Note on the Equation of the Two Planes which can be drawn through Two given Points to touch a Quadric:" Mr. H. M. Taylor, M.A.

In addition to the above, a few minor communications were made to the Society.

The same Journals had been subscribed for, and the same exchanges made, as in the preceding Session.

The Meeting then proceeded to the election of the new Council. The Scrutators (Messrs. J. Hammond and C. Pendlebury) having examined the Balloting Lists, declared the following gentlemen duly elected:—

President, S. Roberts, F.R.S.; Vice-Presidents, Dr. Hirst, F.R.S., J. W. L. Glaisher, F.R.S.; Treasurer, C. W. Merrifield, F.R.S.; Hon. Secretaries, M. Jenkins, M.A., R. Tucker, M.A.; other Members, Prof. Cayley, F.R.S., H. Hart, M.A., Prof. Henrici, F.R.S., Dr. Hopkinson, F.R.S., A. B. Kempe, B.A., R. F. Scott, M.A., Prof. H. J. S. Smith, F.R.S., H. W. Lloyd Tanner, M.A., H. M. Taylor, M.A., and J. J. Walker, M.A.

Mr. Roberts having taken the Chair, Mr. Merrifield proceeded to read his Address, entitled, "Considerations respecting the Translation of Series of Observations into Continuous Formulæ." On the motion of Prof. Cayley, it was unanimously resolved that the Address should be printed in the "Proceedings" of the Society.

The following further communications were made:—

"On Bicircular Quartics, with a Triple and Double Focus, and three Single Foci, all of them Collinear: Mr. H. M. Jeffery, F.R.S.

"Further Remarks on the Geometrical Method of Reversion:" Rev. C. Taylor, M.A.

The following presents were made to the Library:—

"Educational Times," November, 1880.

"Nautical Almanac for the Year 1884:" from the Lords Commissioners of the Admiralty.

"Monatsbericht," August, 1880.

"Crelle's Journal," Band 90, 2es Heft.

"Proceedings of the Manchester Literary and Philosophical Society," Vols. xvi., xvii., xviii., xix. (1877, 1878, 1879, 1880).

"Memoirs of the Literary and Philosophical Society of Manchester," 3rd Series, Vol. vi., 1879.

"Beiblätter zu den Annalen der Physik und Chemie," Leipzig, 1880, Band iv., Stucke 7, 11.

Considerations respecting the Translation of Series of Observations into Continuous Formulæ. By C. W. Merrifield, F.R.S.

[*Read November* 11*th*, 1880.]

It has been the custom of the Presidents of this Society, upon vacating their office, to address to the members a few observations, usually rather general in character, concerning some of the objects for the consideration of which the Society was founded. I am unwilling to break through this good custom, and, although my daily employment leaves me but little leisure for independent investigation, or for reviewing recent progress in mathematics, it may be that I shall not take up your time uselessly if I lay before you a few considerations, which have occurred to me, respecting the Translation of Series of Observations into Continuous Formulæ.

An observation is one of those simple words, each of which represents a group of ideas, the combination of which is very far removed from simplicity. The mere sight of a phenomenon is not an observation. To make it one, it is necessary to connote other phenomena, which must be connected with it by definite numerical relation, in respect of time, or of place, or of some other of the measurable attributes of space or matter. This connotation has always been a sufficiently difficult matter to settle with any degree of exactness; so much so, indeed, that really good observations are difficult to obtain, especially those which are sufficiently exact, and closely enough connected with others, to enable us to establish or to verify formulæ, or to settle constants or units.

Without attempting to go into such deep reasonings as are involved in the ultimate question of whether continuity is an objective or subjective phenomenon, it is not, perhaps, going too far into generalities to say, that our observation of nature tends to discontinuity, and our discussion of it to continuity; and thus there is a tendency, on mere grounds of facility of reasoning, to assume continuity practically, as underlying most connected series of observations. Some of the

particular forms of the abuse of this tendency are what I propose to mention to you this evening.

To fix our ideas, let us consider the variations in the limited population of an island, taking the population as an ordinate, while the abscissa represents time. Every arrival, whether by birth or immigration, gives an abrupt vertical step upwards; every departure, whether by emigration or death, gives an abrupt vertical step downwards, the population line being horizontal from one step to another. However large the population of the island may be, even if it be equal to that of the whole earth, we have actually and ultimately no curve at all, but a zig-zag line with rectangular flexures. When we represent a large population in this fashion by a small scale drawing, we obtain, upon applying any process which has the effect of sinking the detail, what is apparently a continuous curve. But this appearance of continuity postulates defective perception, and has no existence of its own. Any attempt at minute exactness of representation by a continuous law lands us in absolute falsehood; and if we carry the results of continuous calculation to extremes, either by the use of many figures of decimals, or by high orders of differences, we are departing in reality from the precision which we are striving to obtain by an apparent exactness of process.

The entire population of the earth has been roughly estimated at (1,500,000,000) fifteen hundred millions. Taking this as unity, it is clear that, if we go to nine decimal figures, the law of variation must be essentially discontinuous. The best representation by a continuous curve cannot therefore be put as a general question, but must be strictly limited by some controlling condition. It is a definite question, when we ask by what straight line, or by what parabola of the n^{th} order, the variation may be best represented, provided n be not too large; but when n is taken too large, the question is not merely indefinite, but misleading, and wholly inept.

If we take any m points of this broken line and choose a function of m parameters of any selected form, it is a perfectly definite question to determine the values of those m parameters, so that the continuous function shall meet the broken line in those points. But we can go no further, for the actual broken line has but two directions, the horizontal and the vertical, while its continuous representation has ever-varying direction. To make the question definite even to this extent, the form of the function has to be made matter of assumption. In ordinary cases, it is usual and expedient to assume the function to be parabolic, that is to say, of the form

$$y = a + a_1 x + a_2 x^2 + \ldots + a_n x^n,$$

and this leads to the simple application of Finite Differences. It need

hardly be said that this assumption is not by any means generally suitable, or even possible, as is obvious in the case of periodicity, which it fails to represent at all; and in such cases as the ordinary increase or decrease of population, in which the law approaches that of a geometrical series, the interpolation has to be made upon the logarithms of the ordinates instead of upon the ordinates themselves. But the point to which I wish to call attention just now is, not the choice of form, but the consideration that, in a large class of natural phenomena, which it is usual to treat by the application of continuous laws, the attempt to obtain too high a degree of approximation is not a mere difficulty, but an absolute blunder, arising from an oversight of the real character of the problem.

Assuming the representative continuous function to have been suitably selected, it becomes a question of some interest, to know what order of differences should be used,—or, more generally, how many parameters should be left to be determined from the results of observation,—so that the error of interpolation should be fairly comparable with the departure from continuity. I cannot find that this problem has received much attention, even to the extent of being definitely stated, except in the case of quadratures. The course has rather been, among the older calculators generally, and also among later calculators who have not bought their experience, either to go to an enormous elaboration of terms and figures, or else to seek some form of function which shall represent the data almost absolutely.

As an example of the former, I may quote Abraham Sharp, who calculated common logarithms to 61 figures, and the areas of circular segments to 17 figures.* In the introduction to the segment table, he gives coefficients for bisection, as far as the twentieth mean difference, upon which he goes on to remark:—"The Operation, I presume, will rarely, if ever, proceed so far, having observ'd that ordinarily if the twelfth Difference exceed 5 Figures, the remoter Differences will encrease; consequently the Subdivision cannot be perform'd truly by any other method save by Calculation." This abuse of figures is quite common with young pupils applying arithmetic to mechanics or physics, as most of my hearers have probably not failed to observe in the course of their own teaching. It is, however, by no means confined to students. I have seen the results of an examination, in a group of schools attended by from 20 to 150 pupils each, brought to a comparison by percentages to four figures of decimals, and the distinguished mathematician who conducted the examination found serious fault with me for cutting down his decimals, when they came to be printed, to two figures; which, being percentages, were considered by

* See "Geometry improved," by A. S., Philomath (London, 1717, 4to), p. 7.

me to be exactly two figures too many. Again, there exists a table for the comparison of English and metric measures, in rather wide circulation, in which the metre is given in inches, to two integers and six decimals,—this being assuredly a superior limit to any possible accuracy,—while the cubic metre is given to five integers and eighteen decimals. It probably involved less mental effort to perform the multiplication simply, than to reflect that the third decimal place marked the superior limit of accuracy in the comparison of the cubic measures. The practice which I have found best, is to keep as much as possible to the same number of significant figures throughout a calculation. The cases are rare in which this needs to be departed from. It is indeed sometimes convenient to use the figures as if they were exact, and not to curtail them in the arithmetical work, in order to retain such useful verifications as the proof of nines; but that convenience affords no justification for recording a result, carried for mere convenience of checking to more figures than are wanted, so as to affect an impossible exactness. That is quite a different thing from using a figure or two more than the observations justify, so as to make sure of not increasing the probable error of the result by arithmetical inaccuracy.

Some of the recent investigations concerning the ellipsoidal figure of the earth have not unfrequently been interpreted with greater generality than the actualities of the problem really permit. The difference between the greater and lesser equatorial axes is given as nearly 10,000 feet, on absolute lengths of a little under 42 million feet. The departure from a spheroidal figure is therefore only 2,500 feet at its maximum. Now, when we have regard to the actual conformation of the earth's surface—with mountains rising above the sea level to a height of fully ten times this departure from spheroidal figure—it becomes necessary to enquire what particular mean this represents. It is evident that the mean figure obtained with reference to mere external form would by no means correspond with the statical mean obtained by considering the earth at rest, but having regard to local density, and making such assumptions as to the mean distribution of density as would render the problem determinate. Again, a different result would be obtained if we used the ellipsoid which would represent the earth dynamically for any specified purpose. Taken in its proper sense, as a mean sea-level,—that is to say, as the figure which would be given by water in a pipe open to the sea at one end, and brought to every possible latitude and longitude at the other end,—it is of course a definite thing. But when we consider how very small is the difference from a much coarser mean, and how very large compared with it are the local irregularities of the earth's surface, regarded either geometrically or statically, it does not appear that much reliable use can be made of

it for general purposes; but it is unfortunately finding its way into the elementary books, with as much precision of statement, and as little restriction of generality, as the eccentricity.

The principal divisions of the general problem of interpolation, setting aside periodicity, are—

1. When the ordinates are numerous, and fairly close to one another.
2. When, for some reason or other, but few ordinates are available, and these usually separated by wide intervals.

In the former case the approximation is good, subject to the condition, in the general case, of not affecting a very extreme accuracy; and in the case where the law of variation is known to be continuous, the approximation may be pushed on without limit; but this is rather a question of computation from formulæ, than of representation of observations. The true test of accurate correspondence is, that all the forms of interpolation should give concordant results, in this sense, that it should be indifferent whether we interpolate directly to the required result, or whether we interpolate to some function of the observations, and then take the inverse functions. If these results differ materially, either result is only reliable within the limits of that difference. Care must, of course, be taken that the function selected shall not itself introduce an approach to discontinuity, as in the case of a reciprocal or logarithm in the neighbourhood of a zero of the variable function.

Where but few ordinates are available, any attempt at general inferences from them is futile, and not to be mended by elaboration of formula, unless we have some extraneous source of information as to the law which the observations must, with more or less approximation, follow. That is the case with new asteroids and comets. We know, from physical considerations, that their paths in space can differ but little from focal ellipses about the sun. Hence a number of observations, sufficient to determine the parameters of this elliptic motion, render the problem of interpolation determinate. That is not so, when there is no indication of law. In that case we must remember that all that our observations really tell us is the state of the function at the point of observation This is obvious enough when generally stated, but, nevertheless, the caution is not needless, for when curves of stability for ships first came into ordinary use for ships of war—I am sorry to say they are not yet in common use for merchant ships—a suggestion was made that it might be useful to combine the ordinates by Stirling's higher rules, instead of the simpler ones. In the calculations made for the navy, however, the officials of the Constructor's department preferred to use the lower rules, and to select the points of discontinuity for the final ordinates of separate summations; and this was evidently better and sounder work.*

* See the "Transactions of the Institution of Naval Architects," Vol. XII. (for 1871), pp. 96—125, for the methods actually employed in the Admiralty.

In the direct application of Finite Differences, James Stirling rightly lays great stress on the selection of a proper form for interpolation, by means of transformation, if necessary. His words (translated from the Latin) are:—"For interpolation is not to be rashly undertaken; but before beginning the work we must inquire what is the simplest series, upon the interpolation of which that of the proposed series depends. And this preparation is usually absolutely necessary, in order to arrive at neat and definite conclusions."* His own illustration consists in forming a series whose n^{th} term is the product of the n^{th} terms of three other series, and then remarking that it is simpler to interpolate to these three series separately, and to take the product of the results of these separate interpolations, than to interpolate directly in the more complex series. This is evidently meant as an illustration of the wider truth,—that such transformation, as is necessary to make the interpolation as small a matter as possible, should always be adopted. A familiar case of this is the interpolation of the logarithms of numbers differing but little from unity, or of the logarithmic sines or tangents of small angles, in which a transformation is always necessary to exactness.

It is but an extension of this principle in another direction, that when any definitely known law governs the phenomenon, to which the observations relate, either as a whole, or in part, that law should enter into the formulæ of interpolation. Thus, for accurate work, the motion of a projectile subject to gravity is often better treated by interpolating its departure from a parabola, than by direct treatment of its ordinates. This advantage, however, disappears when the departure is excessive, as when the resistance of the medium is very great, or when the trajectory is followed to an extreme length. In these cases a formula must be used which takes account of resistance, and the subject of interpolation or comparison must be the departure from the path thus indicated. This caution is even more necessary to be observed where there is periodicity. When that exists, no interpolation which fails to take notice of it is worth anything if it covers more than a very small fraction of the period.

This consideration emphasizes the importance both of inferring a continuous law from such observations as we can command of any phenomenon suspected to be continuous, and also of testing that law when we have inferred its form. The actual inference of the form is a matter subject to no law. Sometimes it is an obvious deduction from physical considerations; at others it is simply that application of wide experience which is called wisdom; occasionally it seems to involve that deep and instinctive insight into the nature of things, which is called genius. The inference once made, its source is immaterial, except

* See Stirling, "Methodus Differentialis" (London, 1730, 4to), p. 88.

as a matter of history; but there are some points relating to its comparison with the observations which are worthy of our attention.

First in order, I may mention the case in which the phenomena are essentially discontinuous. I have already mentioned one instance, in the law of population of an island. Another is to be found in the formula which represents the number of primes below a limit x, namely,

$$x : (\log_e x - a).$$

There are two things to be observed about this. In the first place, the irregularity in the occurrence of the prime numbers renders any exact formula necessarily untrue, when we go into minute detail. Secondly, the quantity a appears to be a function of x, and not a constant.*

There is a remarkable instance in which an experimental hypothesis was found to represent with very great exactness extensive series of statistics, which, to casual observation, entirely failed to suggest any obvious law. The late Benjamin Gompertz suggested that the law of human mortality might be very nearly expressed upon the hypothesis that the force of vitality diminishes by equal quantities in equal times, reckoned from birth. This leads to the following expression for the table of decrements of life:—

$$dy = -ab^x y\,dx,$$

or

$$y = p^{q^x},$$

where a and b, or p and q, are parameters to be determined from the observations, and y represents the number living at the end of x years. It is found that this form, with different values for the parameters, represents all the ordinary life tables over a very large portion of their extent. Nevertheless, it does not represent the best tables from end to end. If adjusted to the middle portion of the table, it fails to represent either infantile or senile life with exactness. It has been attempted to meet this difficulty by treating these parts of the table discontinuously. This is a very reasonable plan, so far as regards arithmetical accuracy, in any results we may deduce from it. Otherwise it is not very satisfactory, for it leaves in doubt whether this mode of representation is intended to cover want of continuity in the phenomenon, or the inexactness of the formula applied to it. In justice to Mr. Gompertz, it should be stated that he is not responsible for this addition to his theory.

Gompertz's formula, however, may be made to represent, by a proper choice of values for its parameter, the main portion of all the principal tables of the decrements of life, in some cases including infantile or senile life, and even both, with a degree of exactness running very close

* On this subject see Glaisher's "Introduction to the Factor Table for the Fourth Million."

upon the limit of continuity of the tables; for, with the exception of the Registrar-General's tables, which go to millions, most of the ordinary life tables only deal with from 10,000 to 100,000 individuals. In these cases, therefore, minute accuracy, beyond three or four figures, lands us in the discontinuity already mentioned, as arising from our falling back upon the individual unit. There is, therefore, no object in pushing the discussion of the formula too far, especially as all these inductions are open to another source of error. As I have already remarked, it is an interesting, and almost untouched problem, to adjust the degree of interpolation to the limit of continuity.

The inaccuracy of a formula may sometimes be masked, almost completely, by giving an average constant value to some really variable parameter. Thus the observations of the trajectory of a rifle bullet may be represented with very great accuracy on the supposition that the resistance throughout the flight varies simply as the cube of the velocity. As the observations extend from initial velocities considerably exceeding that of a sound wave in air, to a final velocity considerably below that of sound, it is obvious that the same physical law cannot hold, with an unchanged parameter, throughout the whole course; and the inference is, that, instead of having obtained the actual law, we have merely obtained an approximate law, sufficiently close to the real one not to be arithmetically distinguishable from it in the results which fall under our observation. This source of error is too often overlooked in the practical verification of assumed laws which are supposed to govern phenomena.

When we are free to choose form, instead of assigning values to the parameters of an arbitrarily chosen form, like that of parabolic interpolation, it is obvious that a suitably selected form may enable us to represent a series of observations with far greater exactness, and yet with a small number of parameters. In fact, the solution of certain differential equations by the variation of parameters, is little more than a particular application of this principle, with some generalisation of it. The advantage of such selected forms is, that they afford some degree of probability that they are approximations to the law governing the phenomena, as well as arithmetically approximate representations of the observations. But mere arithmetical fitting, obtained by the assumption of complex forms with numerous parameters, is not only no evidence that the true law has been approximately found, but affords no justification for the use of the formula beyond the extremes of the observations. The process is, in fact, of no more value than the application of parabolic interpolation to a series not ascertained to be free from periodicity, or from infinite values of differential coefficients, within the limits; and it is usually more troublesome.

The foregoing remarks are irrespective of any uncertainty as to the

individual observations. When a number of observations of a phenomenon, which can yield but a single numerical value, have to be compared, the ordinary theory of the errors of observation furnishes the most probable numerical amount of that value, or of any given function of that value; and this whether the observations be all equally good, or have definite numerical weights attached to each. A further refinement has been introduced by attaching weights, themselves derived from the departure of the individual observations from the first mean. This is a perfectly definite process, and the only remark which needs to be made upon it here is, that the most probable value of a given function of the result is not the same thing as the given function of the most probable value of the result.

When an unknown curve is only known by a number of points, each determined subject to some unknown but appreciable error, the problem of finding the curve is absolutely indeterminate, unless some assumption be made as to the nature of the curve. This will be best seen by taking an easy problem, in which the indeterminateness is removed by a simple supposition. Let us assume that a right line has been observed, and is to be plotted by means of a set of equidistant ordinates, but that, upon setting them off, the heads are not in a right line. It is then a perfectly definite problem, to find a right line such that the squares of the distances of the points from it shall be a minimum; and, in accordance with the fundamental principle of the ordinary theory, we shall find the same right line in whatever uniform direction we measure the distances. But the assumption that the line through the observed points is a right line, is exactly what we want to avoid in the general problem. On the other hand, when points of a curve are definitely given, we may make the curve determinate by assuming that its continuity is of the highest order possible. In its simplest form this is effected by assuming the curve to have a parabolic equation; but this is not essential, and we may settle it by circular curvature instead of by parabolic order. But, whatever law of facility we take for the nature of the curve connecting the points, that is evidently independent of the law by which the points are assumed, and there is no law connecting the two systems of probability. Any attempt to attain determinateness is therefore of necessity futile.

This indeterminateness is experienced in practice as well as indicated by theory. One of the commonest modes of "fairing" a curve through given points is by using a flexible batten, or spline, which is pinned down by lead weights to the points through which the curve is to be drawn, and the pen is then drawn along the batten. Now, in practice, it is found impossible to use similar battens for all curves. The batten has to be weakest where the curvature is the greatest, and it is a matter of taste and discrimination to select a batten with the proper

taper, and to use it discreetly, so as to get a reasonable and presentable result. The use of moulds or curved patterns is still more a matter of eye.

In the case of a curved surface, such as that of a ship, the problem is rendered somewhat more determinate by the consideration that all the sections, and all their projections, must be fair curves. The two sets of vertical sections, and the water sections, thus correct one another, and it is not an uncommon thing to complete the "fairing" by means of diagonal lines. Another mode, nearly equivalent, is to make a model, and to work it until it is not only quite smooth, but until, when it is held up in every possible light, the shadows fall evenly and fairly upon it. This is quite as severe a test as the drawing. Nevertheless, in either case, the adjustment is not a matter of rule, but of taste and judgment. Apart from the mechanical skill necessary to produce such work, there are many people whose perceptions are not sufficiently delicate to appreciate or test it.

While the problem is thus really and intrinsically indeterminate, all the solutions being strictly *secundum quid*, instead of being general, the difficulty is by no means beyond the reach of practical skill in the most useful cases. A comparatively small number of sections in two dimensions will enable two experienced draughtsmen to produce a couple of ships which shall differ very little in size or shape when they come to be built.

It may be worth while to repeat that the indeterminateness really turns upon the want of any arithmetical comparison between two independent systems of variation of error, or of any analytical means of combining them so as to give a single determinate result.

When the observation, regarded as a point in space, is not only uncertain in linear position, but is uncertain in two or three dimensions, so as to be limited by a circle or sphere of indefinite radius, there is further indeterminateness. This appears, at first sight, to be indeterminate in an independent manner, as being in another dimension. Whether any method can be devised which shall render the errors in different dimensions comparable, I do not know; but I have seen none as yet, and there will still remain the insuperable difficulty about the arithmetical comparison of the probable error of the line or surface through definite points, with the probable error in the position of the points.

Many of the foregoing remarks would seem to apply to the discussion of mean values, so often met with in meteorological works, and especially to means of means, both in respect of what the mean result really represents physically, and also as to whether the processes do not "plane down," or "smooth away," features essential to the theory, as well as local irregularities.

I fear that I have brought before you little that is new, and nothing that would fail to occur to any careful thinker, who gave real attention to the subject. Nevertheless, I hope that you will not think your time wasted, by my thus bringing before you, with some degree of system, a discussion of some of the limits of precision in the application of continuous formulæ to series of observations.

December 9th, 1880.

S. ROBERTS, Esq., F.R.S., President, in the Chair.

Mr. William Ralph Westropp Roberts, M.A., Ex-scholar, and Mr. Ralph Augustus Roberts, B.A., Scholar of Trinity College, Dublin, were elected Members of the Society.

The Auditor (Mr. J. Stirling, M.A.) stated that he had been to the Bank of England and found that the Treasurer's statement as to the monies therein deposited was true, and that he had examined the accounts and found them correct. A vote of thanks was passed to the Auditor on the proposal of the Chairman.

The following communications were made:

"Note sur la Dérivation des Déterminants:" Prof. Teixeira, Coimbra, Portugal.

"Solution of the Binomial Equation $x^p-1=0$, Quinquisection:" Prof. Cayley, F.R.S.

"A General Theorem in Kinematics:" Prof. Minchin.

"On the Solution of the Inverse Logical Problem:" Mr. W. B. Grove, B.A.

"Motion of a Viscous Fluid:" Mr. T. Craig.

"On the Electrical Capacity of a Conductor bounded by two Spherical Surfaces cutting at any angle:" Mr. W. D. Niven, M.A.

The following presents were made to the Library:—

"Atti della Reale Accademia dei Lincei,—Transunti," Vol. v., fasc. 1°, 2°; Roma, 1881.

"Proceedings of Royal Society," Vol. xxxi., No. 207.

"Educational Times," December, 1880.

"Beiblätter zu den Annalen der Physik und Chemie," Band iv., Stück 12.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," 2e série, tome iv., 1er cahier; Paris, 1880.

"Jahrbuch über die Fortschritte der Mathematik," zehnter Band, Jahrgang, 1878, Heft 3; Berlin, 1881.

The Binomial Equation $x^p-1=0$: Quinquisection. By Prof. CAYLEY, F.R.S.

[*Read December 9th*, 1880.]

The theory should be precisely analogous to those for the trisection and quadrisection (see my paper, "The Binomial Equation $x^p-1=0$, Trisection and Quadrisection," *Proceedings*, London Mathematical Society, Vol. xi., 1879, pp. 4—17); only I have not been able to carry it so far. We have in the present case five periods X, Y, Z, W, T, the actual expressions for which, $X=\eta^1+\ldots$, $Y=\eta^3+\ldots$, etc., with Reuschle's selected prime root g, can be (for the primes $5n+1$ under 100) at once written down by means of the table given, pp. 16, 17, of that paper. The relations between the periods are of the form

$$\begin{array}{r|ccccc} & X & Y & Z & W & T \\ \hline X^2 = & a & b & c & d & e \\ XY = & f & g & h & i & j \\ XZ = & k & l & m & n & o; \end{array}$$

that is, we have $\quad X^2=(a,\,b,\,c,\,d,\,e\between X,\,Y,\,Z,\,W,\,T),$

and thence, by cyclical permutations,

$$Y^2=(e,\,a,\,b,\,c,\,d\between \quad\text{,,}\quad), \text{ etc.};$$

viz., from the value of X^2 we have those of Y^2, Z^2, W^2, T^2; from the value of XY those of YZ, ZW, WT, TX; and from the value of XZ those of YW, ZT, WX, TY.

From the equation $X+Y+Z+W+T=-1$, multiplying by X and then substituting for X^2, XY, &c., their values, we obtain

$$\begin{aligned} -a &= 1+f+k+m+g, \\ -b &= \quad g+l+n+k, \\ -c &= \quad h+m+o+i, \\ -d &= \quad i+n+k+j, \\ -e &= \quad j+o+l+f, \end{aligned}$$

which determine (a, b, c, d, e) in terms of (f, g, h, i, j) and (k, l, m, n, o). It is, moreover, easy to prove that

$$\begin{aligned} f+g+h+i+j &= \tfrac{1}{5}(p-1), \\ k+l+m+n+o &= \tfrac{1}{5}(p-1), \end{aligned}$$

whence also $\quad a+b+c+d+e=-1-\tfrac{4}{5}(p-1).$

We obtain other relations between the coefficients by considering the two triple products XYZ and XYW (these are all that need be con-

sidered, since the other triple products are deducible from them by cyclical permutations). From the first of these we have

$$X.YZ = Y.XZ = Z.XW,$$

and from the second $X.YW = Y.XW = W.XY$;

and if we herein substitute for YZ, XZ, &c., their values, and then in the resulting equations for X^2, XY, &c., their values as linear functions of X, Y, Z, W, T, we obtain in all $5.2.2 = 20$ quadric relations between the 15 coefficients; or if we substitute for (a, b, c, d, e) their foregoing values, in all 20 relations between the 10 coefficients (f, g, h, i, j) and (k, l, m, n, o). These are at most equivalent to 8 independent equations, since we have, besides, the sums $f+g+h+i+j$ and $k+l+m+n+o$ each $= \frac{1}{5}(p-1)$; but I have not succeeded in finding the connections between them, or even in ascertaining to how many independent equations they are equivalent.

For any given prime $p = 5n+1$, the values of the coefficients, and also the coefficients of the quintic equation for the periods, could of course be calculated directly from the expressions of the periods; but for the primes under 100, that is, for the values 11, 31, 41, 61, 71, they are at once obtained from Reuschle. We have thus the two Tables, the former giving the coefficients $a, b \ldots n, o$, and the latter the coefficients of the quintic equations.

TABLE 1 OF

p	a / f / k	b / g / l	c / h / m	d / i / n	e / j / o
11	−2	−1	−2	−2	−2
	1	0	0	1	0
	0	0	0	1	1
31	−4	−6	−6	−4	−5
	0	1	2	1	2
	0	2	2	1	1
41	−8	−5	−6	−6	−8
	3	0	2	1	2
	2	2	2	1	1
61	−10	−9	−12	−8	−10
	3	2	2	3	2
	0	2	4	3	3
71	−14	−10	−12	−9	−12
	4	2	3	2	3
	2	3	5	2	2

TABLE 2 OF THE QUINTIC EQUATIONS.

COEFFICIENTS OF

p	η^5	η^4	η^3	η^2	η^1	1	
11	1	1	−4	−3	+3	+1	= 0.
31	1	1	−12	−2	+1	+5	
41	1	1	−16	+5	+21	−9	
61	1	1	−24	−17	+41	−23	
71	1	1	−28	+37	+25	+1	

On Bicircular Quartics, with a Triple and a Double Focus, and Three Single Foci, all of them Collinear. By Henry M. Jeffery, F.R.S.

[*Read Nov.* 11*th*, 1880.]

1. Such a group of quartics with collinear foci may be thus expressed

$$\kappa = (1+dx)(x^2+y^2)+\lambda(x^2+y^2)^2.$$

These bicircular quartics satisfy the relation which connects a point with three single foci $\quad l\rho_1+m\rho_2+n\rho_3=0.$

This property defines l, m, n in terms of a, b, c, the distances of the three single foci from the triple focus or origin.

$$\frac{l\sqrt{a}}{b-c}=\frac{m\sqrt{b}}{c-a}=\frac{n\sqrt{c}}{a-b}.$$

These quartics, if non-singular, are of the eighth class, and have eight foci—a triple, a double, and three single foci. But if the quartics are singular and nodal, two single foci unite at a node and disappear, so that the quartics are of the sixth class.

In the limiting position, an acnode and a crunode unite to form a cusp in a quartic of the fifth class, in which a triple and a double focus remain. Hence, it appears that the preceding property of the focal distances is confined to the single foci, and is inapplicable to these bicircular quartics, if nodal.

2. All the nodal quartics of this group are either of the limaçon type—*i.e.*, are unifolium (defined § 14, and exhibited in Fig. 4) quartics in their nascent state; but do not comprise the true limaçon—or are lemniscatoid, and include the true lemniscata, being bifolium quartics in their nascent state. The companion curves in these cases are respectively unifolium and bifolium. The acnodal quartics are connected with the lemniscatoids by pairs of mere ovals, as the parameters gradually change their values. In all other cases the quartic is a mere oval. These varieties are determined by a mutual relation, which exists between the parameters κ and λ, when the quartics are nodal, and is here exhibited as a quintic discriminating curve. (Fig. 1.)

3. To complete the exposition, a second mutual relation between the parameters should be found, when there are in a group of quartics points of undulation or folium-points, at which quartics pass from

folium to non-folium quartics. This mutual relation is here exhibited as a second discriminating quintic. (Fig. 6.)

---

4. Let the two forms of the quartic be compared, viz.,

$$\kappa = (1+dx)(x^2+y^2)+\lambda(x^2+y^2)^2 \quad \text{..................(A)},$$
$$l\rho_1+m\rho_2+n\rho_3 = 0,$$

where $\rho_1^2 = (x+a)^2+y^2$, and ρ_2, ρ_3 have like values.

The coefficients of x and x^2 in the latter must vanish. Hence

$$l\sqrt{a}+m\sqrt{b}+n\sqrt{c} = 0,$$
$$(l^2a-m^2b-n^2c)(l^2a^2-m^2b^2-n^2c^2)-2m^2n^2bc(b+c) = 0.$$

If l be eliminated,

$$(m\sqrt{b}+n\sqrt{c})\{m\sqrt{b}(a-b)+n\sqrt{c}(a-c)\} = 0.$$

The first factor gives $a = 0$, the triple focus at the origin, and is irrelevant; from the second, $\dfrac{l\sqrt{a}}{b-c} = \dfrac{m\sqrt{b}}{c-a} = \dfrac{n\sqrt{c}}{a-b}$.

5. If these values of l, m, n be substituted, the non-singular quartic is

$$\rho_1\frac{(b-c)}{\sqrt{a}}+\rho_2\frac{(c-a)}{\sqrt{b}}+\rho_3\frac{(a-b)}{\sqrt{c}} = 0,$$

where ρ_1^2, ρ_2^2, ρ_3^2 have the values $(x+a)^2+y^2$......

This may be expanded, and reduced to the form in which the coefficients are functions of the focal distances from the triple focus

$$(a^2+b^2+c^2-2bc-2ca-2ab)(x^2+y^2)^2-2abc(a+b+c+4x)(x^2+y^2)$$
$$+a^2b^2c^2 = 0 \quad \text{............(B)}.$$

Hence $\lambda = -\dfrac{\Sigma(a^2-2bc)}{2abc(a+b+c)}$, $\kappa = \dfrac{abc}{2(a+b+c)}$, $-d = \dfrac{4}{a+b+c}$.

Cor.—If $b = c$, or the two foci unite in a double focus, $l = 0$, and the form fails; so that the fundamental property of bicircular quartics (§ 1) is inapplicable.

6. To determine the foci in bircular quartics of this group.

First, resume the equation

$$\kappa = (1+dx)(x^2+y^2)+\lambda(x^2+y^2)^2.$$

If S and T are the quartic and sextic invariants, which denote the envelopes of lines $(x\xi+y\eta-1 = 0)$ which cut the quartic equiharmonically and harmonically, the tangential equivalent to this quartic is

$$S^3-27T^2 = 0;$$

$$3S=(2\lambda+\xi^2+\eta^2+d\xi)^2-\tfrac{3}{4}(\xi^2+\eta^2)\left[(\xi+d)^2+\eta^2\right]-3\kappa\lambda(\xi^2+\eta^2)^2,$$

$$27T=-(2\lambda+\xi^2+\eta^2+d\xi)^3+\tfrac{9}{8}(2\lambda+\xi^2+\eta^2+d\xi)(\xi^2+\eta^2)\left[(\xi+d)^2+\eta^2\right]$$
$$+9\kappa(\xi^2+\eta^2)\left\{\lambda^2+\frac{\lambda}{2}(\xi^2+\eta^2+d\xi)-\tfrac{3}{16}d^2\eta^2\right\}.$$

Now, if the terms be first considered which do not contain (κ), the expansion of $\left(\dfrac{S}{3}\right)^3-T^2$ gives

$$\tfrac{27}{64}(\xi^2+\eta^2)^2\left[(\xi+d)^2+\eta^2\right]^2\left[4\lambda^2+4\lambda(\xi^2+\eta^2+d\xi)-d^2\eta^2\right].$$

Hence $(\xi^2+\eta^2)^2$ measures the equivalent tangential equation, which is thereby reduced to the eighth class, as was anticipated, since there are at infinity two double points, which will be shown (in § 17) to be flecnodes. The foci are obtained, by rejecting all terms in the equivalent equation, as above reduced, which contain $(\xi^2+\eta^2)$. The remaining terms are

$$\tfrac{27}{64}(d^2+2d\xi)^2(4\lambda^2+4\lambda d\xi-d^2\eta^2)-9\kappa\lambda(2\lambda+d\xi)^4$$
$$+18\kappa(2\lambda+d\xi)^3(\lambda^2+\tfrac{1}{2}\lambda d\xi-\tfrac{3}{16}d^2\eta^2).$$

If, further, $\xi^2-(\xi^2+\eta^2)$ be written for η^2 in this residuum, $(2\lambda+d\xi)^2$ is recognised as the common factor, which accordingly denotes the double focus. The other factor denotes the three single foci

$$(d^2+2d\xi)^2+8\kappa d^2\xi^2(2\lambda+d\xi)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(C).$$

The origin is a triple focus ($1^3=0$), since the curve is a class-octavic, and must have eight foci.

The discriminant of (C) is the discriminant of (A) when $y=0$, or two apses coincide, viz., the quintic

$$27\kappa d^4-4d^2(1+36\kappa\lambda)+16\lambda(1+4\kappa\lambda)^2=0\ldots\ldots\ldots\ldots\ldots(D).$$

Hence it appears (1) that for a nodal curve two foci coincide, and (2) that two of the single foci may be imaginary on the axis of collinearity.

The equation (D) exhibits the mutual relation between the parameters when the two apses coincide at a node, and will be used to distinguish the various curves of the bicircular group. But it shall next be shown, by reference to the other quartic form (B), that two foci coincide at a node, and disappear; this is a theorem proved by Prof. Casey.

The double focus may be obtained in the usual way from the equation directly. It also represents the focal conic of Dr. Casey ("On Bicircular Quartics," § 14, Dublin, 1869.) Since the quartic is (B)

$$\frac{b-c}{\sqrt{a}}r_1+\frac{c-a}{\sqrt{b}}r_2+\frac{a-b}{\sqrt{c}}r_3=0,$$

the focal conic is

$$\frac{b-c}{\sqrt{a}}(1+a\xi)^{\frac{1}{2}}+\frac{c-a}{\sqrt{b}}(1+b\xi)^{\frac{1}{2}}+\frac{a-b}{\sqrt{c}}(1+c\xi)^{\frac{1}{2}}=0.$$

It has been shown (§ 5) that the quartic, when expanded, becomes

$$\lambda r^4 + (1+dx)\, r^2 = \kappa \quad \text{.......................(A).}$$

So the equation to the focal conic is expanded

$$2\lambda + d\xi = 0.$$

7. Next, to determine the conditions of singularity in the quartic

$$\Sigma (a^2 - 2bc)(x^2+y^2)^2 - 2abc(a+b+c+4x)(x^2+y^2) + a^2b^2c^2 = 0 \quad \text{......(B).}$$

In this case two apses coincide, and $y = 0$.

The resulting quartic will have equal roots, if its quartic and sextic invariants are connected by the condition

$$\left(\frac{S}{3}\right)^3 - T^2 = 0,$$

$$[(a+b+c)^2 - 3bc]^3 - [(a+b+c)^3 - \tfrac{9}{2}\Sigma a^2 b]^2 = \tfrac{27}{4}(b-c)^2(c-a)^2(a-b)^2 = 0.$$

Hence, for a critical quartic, two of the three focal distances (a, b, c) must coincide.

8. This condition incidentally gives a solution of the indeterminate equation $\quad x^2 + 27y^2 = z^3, \quad$ or of $\; 3x^2 + y^2 = z^3$, if $(a+b+c) = 3n$, unless a, b, c are in Arithmetical Progression, when $2(a+b+c)^3 = 9\Sigma a^2 b$, and the form would fail.

But a direct solution is given by Euler ("Algebra," p. 399), of the general equation $x^2 + my^2 = z^3$.

$$x = s(s^2 - 3mt^2), \quad y = (3s^2 - mt^2)\, t.$$

This may be identified with the preceding solution, if

$$c - b = 3t + s, \quad b - a = 3t - s.$$

Euler has tabulated the lowest solutions of the particular form

$$3x^2 + y^2 = z^3.$$

9. If the curve be nodal, the quartic (B) becomes

$$(a-4b)(x^2+y^2)^2 - 2b^2(a+2b+4x)(x^2+y^2) + ab^4 = 0.$$

Let the origin be transferred from the triple focus to the coincident foci $(b = c)$, *i.e.*, let $x = x + b$; the transformed equation shows that the node is at this point

$$\left(\frac{a}{4} - b\right)(x^2+y^2)^2 + b\,[(a-2b)\,x - b^2]\,(x^2+y^2) + ab^2x^2 = 0 \quad \text{.....(E).}$$

The several nodal quartics can be conveniently distinguished by this form, but as it does not embrace the companion-curves, which are non-singular, the quartic form (A) is resumed.

10. To exhibit the mutual relation which exists between the parameters κ and λ, when the bicircular quartics in the group under consideration are singular, as the first discriminating curve.

Resuming the quartic

$$\kappa = (1+dx)(x^2+y^2)+\lambda(x^2+y^2)^2 \;\dots\dots\dots\dots\dots\dots \text{(A)},$$

the conditions of singularity are

$$(1)\;\; 4\kappa = 2x^2+dx^3,\;\; y=0,$$
$$(2)\;\; 4\lambda x^2+3dx+2=0.$$

Their eliminant is otherwise obtained in § 6,

$$27\kappa d^4-4d^2(1+36\kappa\lambda)+16\lambda(1+4\kappa\lambda)^2 = 0 \;\dots\dots\dots\dots\dots \text{(D)}.$$

But the curve is conveniently drawn, and its properties examined, by the implicit equations (1) and (2).

At a singular point on the quintic $\frac{d\kappa}{dx}=0$, $\frac{d\lambda}{dx}=0$; there are thus two cusps, one at infinity, when $x=0$ at the extremity of the (λ) axis of co-ordinates; and another, when $4+3dx=0$, $27d^2\kappa=8$, $32\lambda=9d^2$. The axes of co-ordinates are the asymptotes, and there are no points of inflexion.

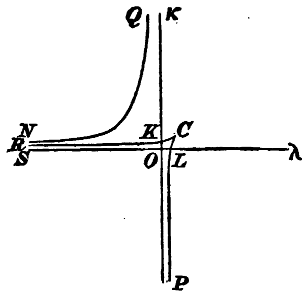


Fig. 1.

The locus of the cusp in all quintics (D) for a family of these groups of bicircular quartics is a rectangular hyperbola.

If d be eliminated from the equations

$$27d^2\kappa=8,\;\; 32\lambda=9d^2,$$

the eliminant is $$12\kappa\lambda=1.$$

11. By the aid of this discriminating quintic, all bicircular quartics of this group may be exhibited.

The general equation (E) to the nodal quartic in § 9 will be conveniently used, as well as (D). The constants in the two equations are thus related: $-d=\frac{4}{a+2b}$, $\kappa=\frac{ab^2}{2(a+2b)}$, $-\lambda=\frac{a-4b}{2b^2(a+2b)}$.

In the first quadrant (Fig. 1), if the point (κ, λ) lies on the portion KC, *i.e.*, if the parameters in the quartic (A) are so connected, the quartics are lemniscatoid; *i.e.*, are nodal, but not biflecnodal, while $a>b<4b$; but if $a=2b$ in (E), or if $d=\frac{1}{b}$, $\kappa=\frac{1}{4d^2}$, $\lambda=\frac{d^2}{4}$ in (A), the quartic is a lemniscata with a biflecnode

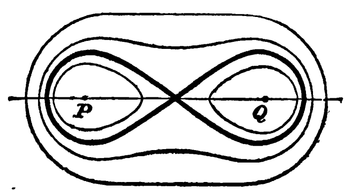


Fig. 2.

$$(x^2+y^2)^2 = 2b^2(x^2-y^2).$$

The companion-curves to the lemniscata, here drawn for higher values of κ, are a bifolium quartic, which passes through another with two folium points (§ 14, Fig. 6) into a pure oval, and for lower values of κ into two ovals, with or without folia, according as (κ, λ) lies beyond or within the second discriminating curve (Fig. 6).

If κ, λ be on the portion LC, and $a < b > 0$, the quartic has an acnode, and, for positions of (κ, λ) beyond the second curve (Fig. 6), becomes a mere oval. At the cusp C of the quintic, this acnode unites with the preceding crunode of a lemniscatoid to form a cusp in the quartic (Fig. 3). In (E), $a = b$, and the only cusped quartic is thus defined,

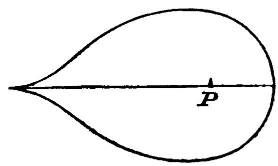


FIG. 3.

$$3(x^2+y^2)^2+4bx(x^2+y^2)+4b^2y^2 = 0.$$

If (κ, λ) is on the axis of κ, $\lambda = 0$, $a = 4b$, the quartic degenerates into the line at infinity and the cubic

$$\kappa = (1+dx)(x^2+y^2),$$

including the nodal curve of form (E), $(2x-b)(x^2+y^2)+4bx^2 = 0$.

These circular cubics are Newton's defective hyperbolæ, with a diameter, species 39, 41, 45; the ampullate cubic represents the folium of the unipartite quartic, and the campaniform the bipartite quartic. (Fig. 5.)

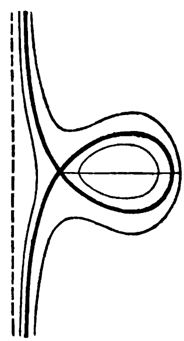


FIG 5.

In the second quadrant the bounding quintic has two branches. If $a+2b < 0$, no nodal quartic is possible. In other words, if (κ, λ) is on or beyond the outer branch NQ of the quintic, no quartic can be drawn. If $a > 4b$, or the point (κ, λ) is on the inner branch RK, the nodal curves are of the limaçon type. But they do not comprise the true limaçon, for the form (E) cannot assume the shape $(x^2+y^2+px)^2 = m^2(x^2+y^2)$.

One companion-curve is unifolium; the other consists of two non-folium ovals, as (κ, λ) lies above or below RK. If, (κ, λ) is on the axis of λ, $\kappa = 0$, the quartic degenerates into a circle and a point circle at the triple-focus. (Fig. 4.)

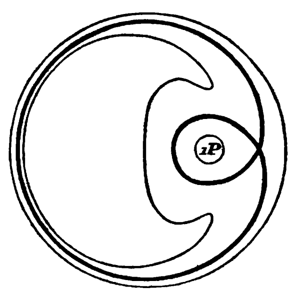


FIG. 4.

In the third quadrant, κ, λ are both negative, there is no critical value, and

the quartics are non-folium single ovals. If (κ, λ) is on the axis of κ and negative, the quartic degenerates as before into a circular cubic, and the line at infinity, but the cubic is campaniform or conchoidal only (Newton's, Fig. 49).

In the fourth quadrant, if (κ, λ) lie between the part LP of the quintic and its asymptote, the quartic is a mere oval. If (κ, λ) is on LP, a, b have opposite signs, but $a < 2b$; then no quartic is possible, nor is any possible if (κ, λ) lie beyond LP. See below § 15.

12. If a bicircular quartic of this group be nodal, two foci, which unite at a node (§ 9), disappear. This proposition by Prof. Casey ("Bicircular Quartics," § 44) will be established for the special case of the lemniscata, and similarly proved to be generally true. In equation (E), if $a = b$, $(x^2+y^2)^2 = 2b^2(x^2-y^2)$.

S and T, defined in § 6, have these values:

$$12S = \left(\xi^2-\eta^2-\frac{2}{b^2}\right)^2,$$

$$216T = \left(\xi^2-\eta^2-\frac{2}{b^2}\right)^3+\frac{27}{b^2}(\xi^2+\eta^2)^2.$$

The tangential equation to the lemniscata is

$$\left(\frac{S}{3}\right)^3-T^2=0,$$

or

$$\left(\xi^2-\eta^2-\frac{2}{b^2}\right)^3+\frac{27}{2b^2}(\xi^2+\eta^2)^2=0.$$

Its geometrical interpretation is

$$(2qr+b^2)^3 = \tfrac{27}{2}\, b^4p^2,$$

if q, r, p are the perpendiculars on a tangent from the triple foci and node of the lemniscata.

There are therefore two triple foci $(\pm b, 0)$. One of these $(-b, 0)$ is the original triple-focus (since the origin was transferred in § 9), and at the other triple-focus the single-focus has united with the double-focus denoted by $(2\lambda+d\xi)^2$ in § 6. Hence the two single foci, which united at the node, have disappeared.

13. The cusped quartic of this group is a class-quintic, and also retains its triple and double foci. (Fig. 3.)

Its equation, as given in § 11, when the three single foci are made coincident, is $$3(x^2+y^2)^2+4bx(x^2+y^2)+4b^2y^2=0.$$

The invariants S and T have the subjoined values:

$$\frac{S}{12} = (1+b\xi)^2\left(1-\frac{b\xi}{3}\right)^2 + \frac{2b^2}{3}(1+b\xi)(\xi^2+\eta^2),$$

$$\frac{T}{8} = (1+b\xi)^3\left(1-\frac{b\xi}{3}\right)^3 + b^2(1+b\xi)^2\left(1-\frac{b\xi}{3}\right)(\xi^2+\eta^2) - \frac{b^4}{2}(\xi^2+\eta^2)^2.$$

For the equivalent tangential equation,

$$S^3 = 27T^2.$$

When reduced, this may assume the form

$$\frac{b^4}{4}(\xi^2+\eta^2)^2 - \frac{b^2}{27}(35-b\xi)\ (1+b\xi)^2\ (\xi^2+\eta^2) - \tfrac{4}{3}(1+b\xi)^3\left(1-\frac{b\xi}{3}\right)^2 = 0.$$

The original origin $(-b, 0)$ is still a triple focus, and $\left(\frac{b}{3}, 0\right)$ is a double focus, as defined in § 6.

This is another proposition by Dr. Casey (§ 44), that if a bicircular quartic has a cusp, it arises from the union of three single foci.

Cor.—The inflexions in this pirum or cusped quartic may be thus determined.

When $S = 0$, $T = 0$, the tangent touches it at a point of inflexion,

$$9-3b\xi+4+4b\xi = 0, \quad b\xi+13 = 0,$$
$$512 = b^2(\xi^2+\eta^2), \quad b\eta = \pm 7\sqrt{7}.$$

14. To determine when folium points disappear in a group of quartics, three methods may be adopted.

Two points of inflexion, as well as a bitangent, characterise the depression in a curve, which Dr. Zeuthen (*Mathematische Annalen*, 1874) designates "*folium.*" Hence, first, the Hessian must touch the quartic at the folium-point, in which the two inflexions unite in the nascent state.

Ex.—The bifolium-companion to the lemniscata (Fig. 2)

$$(x^2+y^2)^2 = 2c^2(3x^2+y^2)+3c^4.$$

Its Hessian is

$$(x^2+y^2)^2(3x^2+y^2)+c^2(15x^4+30x^2y^2+7y^4)-3c^4(9x^2+11y^2)+9c^6 = 0.$$

Secondly, at folium-points, both $S = 0$ and $T = 0$.

Thirdly, $\qquad \dfrac{d^2y}{dx^2} = 0$ and $\dfrac{d^3y}{dx^3} = 0;$

or if polar coordinates be used, and $ru = 1$,

$$\frac{d^2u}{d\theta^2} + u = 0, \quad \frac{d^3u}{d\theta^3} + \frac{du}{d\theta} = 0.$$

The two curves touch at the points $(0, \pm c\sqrt{3})$.

15. To exhibit the mutual relation which exists between the parameters, when there are folium-points in this group of bicircular quartics, as a second discriminating curve. (Fig. 6.)

Let the general equation (A) be resumed,

$$\kappa = (1-dx)\,(x^2+y^2)+\lambda\,(x^2+y^2)^2.$$

Differentiate thrice; then the conditions of § 13 $\left(\dfrac{d^2y}{dx^2} = 0 = \dfrac{d^3y}{dx^3}\right)$ will yield the relations,

$$1-dx = 0,$$

$$d = 4\lambda\left(x+y\,\frac{dy}{dx}\right),$$

$$d^2 = 8\lambda^2\,(x^2+y^2)\left\{1+\left(\frac{dy}{dx}\right)^2\right\}.$$

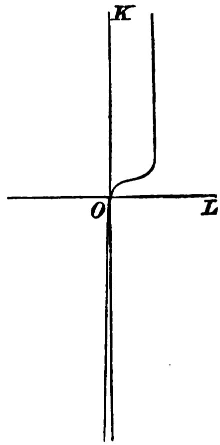


Fig. 6.

The eliminant is a quintic, which is the second discriminating curve.

$$\kappa\,(d^2+8\lambda)^2 = 4\lambda\,(8\kappa\lambda+1)^2.$$

If polar coordinates be used, the equation to the group is

$$\cos\theta = \lambda r+\frac{1}{r}-\frac{\kappa}{r^3}.$$

The conditions of § 13 yield the relations from which the quintic may be conveniently drawn as a unicursal curve,

$$\left(4\lambda-\frac{1}{r^2}\right)^2 = \frac{d^2}{r^2}-\frac{1}{r^4},\quad \kappa = \lambda r^4.$$

If κ, λ are both negative, $\cos\theta$ cannot have a maximum or minimum value; hence, for values of (κ, λ) in the third quadrant, there is no folium.

The relation $(\kappa = \lambda r^4)$ shows that no values of (κ, λ) are in the second and fourth quadrants in Fig. 6.

This quintic has three inflexions, and the axis of (κ) for its asymptote. It intersects the first discriminating curve (Fig. 1, § 10), when $4\kappa d^2 = 1$, $4\lambda = d^2$; this occurs when the corresponding quartic is the lemniscata (§ 12). It has been shown to give real folium-points only when (λ, κ) is in the first quadrant.

16. Bicircular quartics alone of this group do not admit of four single collinear foci.

All these quartics with collinear foci satisfy the fundamental relation

$$l\rho_1+m\rho_2+n\rho_3 = 0,$$

where the focal distances are ρ_1, ρ_2, ρ_3, as in § 5.

The conditions are two for a fourth single focus or point-circle

$$\lambda\rho_1^2+\mu\rho_2^2+\nu\rho_3^2 = 0,$$

(1) of tangency $$\frac{l^2}{\lambda}+\frac{m^2}{\mu}+\frac{n^2}{\nu}=0,$$

(2) the discriminant, which determines the focus as the intersection of two imaginary straight lines

$$\frac{(b-c)^2}{\lambda}+\frac{(c-a)^2}{\mu}+\frac{(a-b)^2}{\nu}=0.$$

These conditions suffice to determine a fourth focus, unless

$$\frac{l\sqrt{a}}{b-c}=\frac{m\sqrt{b}}{c-a}=\frac{n\sqrt{c}}{a-b}, \text{ as in § 4};$$

whence $$\frac{\lambda a}{b-c}=\frac{\mu b}{c-a}=\frac{\nu c}{a-b}.$$

In this case, $$\lambda a+\mu b+\nu c=0,$$ $$\lambda a^2+\mu b^2+\nu c^2=0,$$

and the point-circle becomes the origin $(x^2+y^2=0)$, or is merged in the triple focus.

17. The triple and double foci in each quartic of the group indicate that the tangents at the two nodes at infinity are inflexional and ordinary; hence they are flecnodes; but the lemniscata is triply biflecnodal, since it has two triple foci.

18. The classification of this group would not be altered, if the parameters were differently chosen.

Let the group be otherwise denoted,

$$1=\mu(1+dx)(x^2+y^2)+\nu(x^2+y^2)^2.$$

By comparing the coefficients with those of the former equation,

$$\mu=\frac{1}{\kappa},\quad \nu=\frac{\lambda}{\kappa}.$$

These are Newton's formulæ for homographic transformation of a curve (*Principia*, Bk. I., Lemma 22). In the quintic thus transformed, the critic values would not be affected.

19. The fundamental property $(l\rho_1+m\rho_2+n\rho_3=0)$ of this group is equally applicable to the dual group of class-quartics with quadruple foci $$\kappa=(1+d\xi)(\xi^2+\eta^2)+\lambda(\xi^2+\eta^2)^2.$$

The distances ρ_1, ρ_2, ρ_3 must now be measured from the foot of the perpendicular on any tangent to three fixed points collinear with the quadruple focus.

In fact, this memoir was written while the author was studying the classification of class-quartics with quadruple foci, which, it is hoped, will shortly find a place in the *Quarterly Journal of Mathematics.*

On the Electrical Capacity of a Conductor bounded by Two Spherical Surfaces cutting at any angle. By W. D. NIVEN, Fellow of Trinity College, Cambridge.

[*Read Dec. 9th*, 1880.]

1. In Clerk-Maxwell's "Electricity and Magnetism," I. § 165, there is given a method of finding the distribution of electricity on a conductor formed by the outer surfaces of two spheres cutting at any angle which is a sub-multiple of two right angles. The method is exceedingly simple, but limited in its range, being applicable only to spheres cutting at one of the angles just described.

The object of this paper is to explain another method, which, though less simple, is applicable to the general case of two spheres cutting at any angle, and to calculate the electrical capacities in a few particular instances.

It may be interesting, however, to examine at the outset why it is that the solution above referred to should be limited in its application. That solution may be presented in the following form:—

2. Let n equal positively electrified points be placed symmetrically round the circumference of a circle at the points $A_1, A_2 \dots A_n$. Let also n negatively electrified points, equal in intensity to the former, be also ranged symmetrically round the circumference at the points $B_1, B_2 \dots B_n$. Then, if A_1B_1 is less than A_1A_2, all the planes drawn through the centre of the circle, perpendicular to its plane and bisecting the arcs A_1B_1, B_1A_2, A_2B_2, &c., are obviously at potential zero. There will be n such planes inclined to one another successively at angles each equal to $\pi \div n$. Any two of the planes may be regarded as forming a double wedge, the half of which, a single wedge, may be taken to be a conductor at zero potential. If the angle of this conductor be $p\pi \div n$, there will be p electrified points inside of it, some of which are positive and some negative. These points being real charges, the remaining $2n-p$ points will be imaginary; but the potential due to the p real charges, and to the electricity induced by them in the conductor, will be for all points in the interior angle of the conductor the same as that due to the whole system of $2n$ points. In like manner, if the $2n-p$ points be real charges, the p points will be imaginary, but will represent for all points in the exterior angle of the conductor the electricity induced upon the conductor. Let $p = 1$, and let B_1 be the point in the interior angle of the conductor at which there is a real electrified point $-UR$; then, when we invert with regard to the point B_1, the radius of inversion being R, we obtain the electrification of a conductor freely charged to potential U, the conductor having spherical faces cutting at

an angle $\pi \div n$. If p be not equal to unity, we obtain by inversion a conductor at potential U under the influence of $p-1$ electrified points, placed in the prolongation of its axis one way or the other, the faces now cutting at an angle $p\pi \div n$.

3. It is therefore clear that the method of point images will not solve the problem of the freely charged conductor, excepting for special cases. The following method was employed by the author in a paper on the "Theory of Electric Images," which was published in the *Proceedings* of this Society, viii. p. 64. For the sake of convenience, the general principles to be made use of may be restated.

If A and B be two charged conductors in presence of one another, their charges being in electrical equilibrium, then (1) the potential at a point inside of A is simply that due to the distributions on A and on B; (2) the potential at any point outside of A, due to the electricity on A only, is that due to a free distribution on A which would raise it to its existing potential, B being supposed out of the field, plus the potential due to the electricity which would be induced on A by the existing electrification of B supposed rigid if A were at zero potential. The latter potential, by an extension of the meaning of the word image, may be referred to as the potential of the negative image of B in A.

Any one may satisfy himself of the truth of the second of these propositions, by reflecting on the simple case of a charged insulated spherical conductor with electrified points outside of it.

The truth of the propositions is moreover not limited to the case of two conductors, but, with a slight modification of the statement, will apply to any number.

4. Let us now apply the above rules in the case of two planes OA, OB at zero potential, and cutting at an angle γ with an electrified point placed at a point C between them. The plane of the paper is supposed to contain the point C, and to be orthogonal to the two planes.

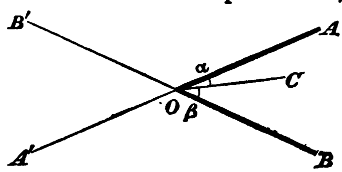


FIG. 1.

Let us put $OC = R$, $C\widehat{O}A = \alpha$, $C\widehat{O}B = \beta$, so that $\gamma = \alpha+\beta$. Any point in the plane of the paper will be given by $OP = r$ and $P\widehat{O}A = \theta$. It will not be necessary to consider points out of the plane of the paper. The potential at P dne to the electrification on OA may be denoted by $\phi(r, \theta)$, and it is clear that θ may be measured at either side of OA, and in like manner the potential at P due to the electrification of OB may be put in the form $\chi(r, P\widehat{O}B)$.

The magnitude of the electrified point may be taken $= -UR$, so that, upon inverting with regard to the point C, the radius of inversion being R, we may have the potential of the new conductor $= U$.

We may obviously treat OA and OB as two separate conductors, for the purpose of applying our fundamental rules. The regions, which are to be regarded as outside and inside of the two conductors, are then those which do and do not respectively contain the electrified point.

By the application of the rules, we have the following equations:—

(α.) In the region AOB', which is inside of A and outside of B,

$$\phi(r,\theta)+\chi(r,\theta+\gamma)-\frac{UR}{\sqrt{R^2-2Rr\cos(\theta+\alpha)+r^2}}=0 \ldots\ldots\ldots\ldots (1),$$

$$\phi(r,\theta+2\gamma)+\chi(r,\theta+\gamma)-\frac{UR}{\sqrt{R^2-2Rr\cos(\theta+\alpha+2\beta)+r^2}}$$

= the potential due to the distribution on B raised to its existing potential, in this case zero, $= 0$ .... (2).

(β.) In the region BOA', which we reckon inside of B and outside of A,

$$\phi(r,\theta)+\chi(r,\theta-\gamma)-\frac{UR}{\sqrt{R^2-2Rr\cos(\theta-\alpha)+r^2}}=0 \ldots\ldots\ldots\ldots (3),$$

$$\phi(r,\theta)+\chi(r,\theta+\gamma)-\frac{UR}{\sqrt{R^2-2Rr\cos(\theta+\alpha)+r^2}}=0 \ldots\ldots\ldots (4).$$

Similar equations may be written down for the other regions, but they will not be wanted.

5. By the subtraction of (2) from (1), we obtain a functional equation for the determination of $\phi(r,\theta)$; viz., we have

$$\phi(r,\theta)-\phi(r,\theta+2\gamma)$$
$$= UR\,(R^2-2rR\cos\overline{\theta+\alpha}+r^2)^{-\frac{1}{2}}-UR\,(R^2-2rR\cos\overline{\theta+\alpha+2\beta}+r^2)^{-\frac{1}{2}}.$$

In like manner, we find

$$\chi(r,\theta-\gamma)-\chi(r,\theta+\gamma)$$
$$= UR\,(R^2-2rR\cos\overline{\theta-\alpha}+r^2)^{-\frac{1}{2}}-UR\,(R^2-2rR\cos\overline{\theta+\alpha}+r^2)^{-\frac{1}{2}}.$$

Or, if we put $\theta-\gamma=\theta'$, the function for χ will take a form similar to that for ϕ; viz.,

$$\chi(r,\theta')-\chi(r,\theta'+2\gamma)$$
$$= UR\,(R^2-2rR\cos\overline{\theta'+\beta}+r^2)^{-\frac{1}{2}}-UR\,(R^2-2rR\cos\overline{\theta'+\beta+2\alpha}+r^2)^{-\frac{1}{2}}.$$

These functional equations cannot in general be solved, but it is easy to show that they are consistent with the solution already obtained, when $n\gamma = \pi$. For the sake of simplicity, let us consider only three positive and three negative points, and let one of the negative points B_1 be the inducing point in the interior angle of the conductor.

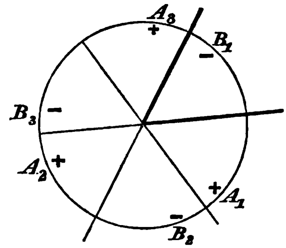


FIG 2.

The potential due to the induced electricity at any point of the interior is

$\phi(r, \theta) + \chi(r, \theta'), = F(r, \theta)$ suppose;

observing that, if θ be increased by 2γ, θ' is diminished by the same amount, we have to determine $F(r, \theta)$ from the equation

$$F(r, \theta) - F(r, \theta + 2\gamma)$$

$$= UR\left[R^2 - 2Rr\cos(\theta - \alpha) + r^2\right]^{-\frac{1}{2}} - UR\left[R^2 - 2Rr\cos(\theta + \alpha + 2\beta) + r^2\right]^{-\frac{1}{2}}$$

$$= UR\left(\frac{1}{PB_1} - \frac{1}{PB_3}\right).$$

This equation is readily seen to be satisfied by

$$F(r, \theta) = UR\left(\frac{1}{PA_1} + \frac{1}{PA_2} + \frac{1}{PA_3} - \frac{1}{PB_2} - \frac{1}{PB_3}\right).$$

The first three terms, taken together, of themselves satisfy the equation, wherever the three points A_1, A_2, A_3 may be. It is only, however, for the particular points in the figure that the further condition, which has been lost sight of by the combination of ϕ and χ, is satisfied; viz.,

that $$F(r, \theta) - UR\frac{1}{PB_1}$$

should vanish at the two faces of the conductor. If we could have solved the equations for ϕ and χ separately in a similar form, the group of three terms referred to would doubtless have appeared in the result naturally, since the fundamental equations (1) and (3) express that the potential is zero at the back of the conductors.

In like manner, when the planes cut at an angle $\pi \div n$, it is easy to show that the solution of the $2n-1$ point-images satisfies the functional equation.

6. Reverting to the functional equations of the last article, let us now make the supposition that they have been solved; we next proceed to find the corresponding solution in the inverted figure. According to the results obtained in the paper already referred to in § 3, it appears that the expression corresponding to $\phi(r, \theta)$ is

$$\frac{R}{CP}\phi\left(R\frac{CP}{OP}, \widehat{CPO} - \alpha\right).$$

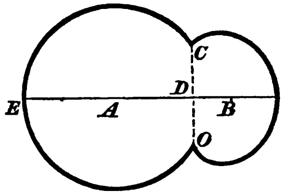


Fig. 3.

Now, the two planes invert into the spheres A and B, and the expression just written down would be the potential at any point P inside of the sphere A, due to the electrification of A only.

7. Although we cannot in general solve the functional equation for ϕ, we can solve it in the particular case when $r = R$ (Fig. 1); that is to say, when we take those points only which lie on the circle whose centre is O and radius OC. The corresponding inverted potential will then be $\dfrac{R}{CP}\,\phi\,(R,\ C\widehat{P}O-\alpha)$,

and this will represent the potential at any point P on part of the axis ED in Fig. 3, due to A's electrification only. Taking into account the equations for the fourth region AOA' (Fig. 1), we see that the point P may be anywhere on the diameter of A.

If we put $\phi\,(R,\,\theta) = f(\theta)$, the functional equation obtained by the subtraction of (2) from (1) is then

$$f(\theta)-f(\theta+2\gamma) = \tfrac{1}{2}U\left\{\operatorname{cosec}\tfrac{1}{2}(\theta+\alpha)-\operatorname{cosec}\tfrac{1}{2}(\theta+\alpha+2\beta)\right\}\ldots(5).$$

But $$\operatorname{cosec}\tfrac{1}{2}(\theta+\alpha) = \cot\tfrac{1}{4}(\theta+\alpha)-\cot\tfrac{1}{2}(\theta+\alpha),$$

and $$\cot\frac{1}{n}\phi = \frac{n}{\phi}+\frac{n}{\phi+n\pi}+\frac{n}{\phi+2n\pi}+\text{\&c.}$$
$$+\frac{n}{\phi-n\pi}+\frac{n}{\phi-2n\pi}+\text{\&c.}$$

Also, the solution of the equation

$$f(\theta)-f(\theta+2\gamma) = \tfrac{1}{2}U(\theta+\kappa)^{-1}$$

is $$f(\theta) = \tfrac{1}{4}\frac{U}{\gamma}\int_0^1 \frac{t^{\frac{\theta+\kappa}{2\gamma}-1}}{1-t}\,dt.$$

These transformations and solutions permit us to write, as the solution of (5), the following:

$$f(\theta) = \tfrac{1}{2}\,\frac{U}{\gamma}\left\{\begin{aligned} &2\int_0^1 \frac{t^{\frac{\theta+\alpha}{2\gamma}-1}\left(1-t^{\frac{\beta}{\gamma}}\right)}{(1-t)\left(1-t^{\frac{2\pi}{\gamma}}\right)}\,dt \\ &+2\int_0^1 \frac{t^{-\frac{\theta+\alpha}{2\gamma}+\frac{2\pi}{\gamma}}\left(1-t^{-\frac{\beta}{\gamma}}\right)}{(1-t)\left(1-t^{\frac{2\pi}{\gamma}}\right)}\,dt \\ &-\int_0^1 \frac{t^{\frac{\theta+\alpha}{2\gamma}-1}\left(1-t^{\frac{\beta}{\gamma}}\right)}{(1-t)\left(1-t^{\frac{\pi}{\gamma}}\right)}\,dt \\ &-\int_0^1 \frac{t^{-\frac{\theta+\alpha}{2\gamma}+\frac{\pi}{\gamma}}\left(1-t^{-\frac{\beta}{\gamma}}\right)}{(1-t)\left(1-t^{\frac{\pi}{\gamma}}\right)}\,dt. \end{aligned}\right.$$

No complementary function is necessary, since, when $U=0$, $f(\theta)=0$.

That the equations of § 4 are sufficient for the complete determination of the problem under consideration, may be seen from the fact that $\phi(r, \theta)$ obviously satisfies Laplace's equation, so also does $\chi(r, \theta')$, and the functions are so chosen that the potential is zero at the back of the conductor. To verify that the second condition is actually satisfied by the particular solution of this article, it is only necessary to write down the values of $\phi(r, 0)$ and $\chi(r, \gamma)$, which last may be found from $\phi(r, \gamma)$, by merely interchanging therein the angles α and β, and to add them together. The potential at the conductor OA at distance R from 0 is then found, on reduction, to be

$$\frac{1}{2}\frac{U}{\gamma}\left\{2\int_0^1 \frac{t^{\frac{\alpha}{2\gamma}-1}}{1-t^{\frac{2\pi}{\gamma}}}\,dt-2\int_0^1 \frac{t^{\frac{2\pi}{\gamma}-\frac{\alpha}{2\gamma}-1}}{1-t^{\frac{2\pi}{\gamma}}}\,dt\right.$$

$$\left.-\int_0^1 \frac{t^{\frac{\alpha}{2\gamma}-1}}{1-t^{\frac{\pi}{\gamma}}}\,dt+\int_0^1 \frac{t^{\frac{\pi}{\gamma}-\frac{\alpha}{2\gamma}-1}}{1-t^{\frac{\pi}{\gamma}}}\,dt\right\}$$

$$-\tfrac{1}{2}U\operatorname{cosec}\tfrac{1}{2}\alpha=0.$$

8. In accordance with the directions contained in §§ 6, 7, the potential at any point P in the diameter of A, due to A's electrification, is to be found by changing the angle $\theta+\alpha$, in the expression just obtained, into $C\widehat{P}O$, and multiplying the result by $R \div CP$. We thus see that the potential referred to is, *cœteris paribus*, a function of P's distance from A's centre only, $\psi(x)$ suppose. Having found $\psi(x)$, we may deduce the potential at any point (x, y) in the plane of the paper due to A's electrification only; viz., by the theory of expansions in Legendre's coefficients, taking one of Jacobi's forms, we have

$$\frac{1}{\pi}\int_0^\pi \psi\left(x+\sqrt{-1}\,y\cos\phi\right)d\phi,$$

or

$$\frac{1}{\pi}\,\frac{a}{\sqrt{x^2+y^2}}\int_0^\pi \psi\left(\frac{a^2\left(x+\sqrt{-1}\,y\cos\phi\right)}{x^2+y^2}\right)d\phi,$$

according as the point considered is inside or outside of A.

The solution of the functional equation (5), together with a similar solution for the sphere B, gives accordingly the potential at all points of space exterior to the charged conductor.

9. *Expression for E_a.*—If we put $C\widehat{P}O=2\alpha$, we get the value of the potential at the centre of A, due to its own electricity. To determine the entire quantity of electricity E_a on A alone, we have to multiply this potential by the radius of A, which we shall denote by a.

Observing that $R = CO = 2a\sin\alpha$, we thus find

$$E_a = Ua\,\frac{\sin\alpha}{\gamma}\left\{\begin{aligned} & 2\int_0^1 \frac{t^{\frac{\alpha}{\gamma}-1}-1}{(1-t)\,(1-t^{\frac{2\pi}{\gamma}})}\,dt \\ & +2\int_0^1 \frac{t^{\frac{2\pi}{\gamma}-1}\,(t^{\frac{\beta}{\gamma}}-1)}{(1-t)\,(1-t^{\frac{2\pi}{\gamma}})}\,dt \\ & -\int_0^1 \frac{t^{\frac{\alpha}{\gamma}-1}-1}{(1-t)\,(1-t^{\frac{\pi}{\gamma}})}\,dt \\ & -\int_0^1 \frac{t^{\frac{\pi}{\gamma}-1}\,(t^{\frac{\beta}{\gamma}}-1)}{(1-t)\,(1-t^{\frac{\pi}{\gamma}})}\,dt. \end{aligned}\right.$$

10. *Expression for $E_a - E_b$.*— The value of E_b may be found from E_a by mere interchange of symbols. We then obtain

$$\begin{aligned} E_a - E_b &= \frac{UR}{2\gamma}\int_0^1 \frac{t^{\frac{\alpha}{\gamma}-1}-t^{-\frac{\alpha}{\gamma}}}{1-t}\,dt \\ &= \tfrac{1}{2}UR\,\frac{\pi}{\gamma}\cot\left(\frac{\alpha}{\gamma}\,\pi\right) \\ &= Ua\sin\alpha\,\frac{\pi}{\gamma}\cot\left(\frac{\alpha}{\gamma}\,\pi\right). \end{aligned}$$

For example, if $\gamma = \frac{1}{2}\pi$,

$$\begin{aligned} E_a - E_b &= Ua\left(\cos\alpha - \frac{\sin^2\alpha}{\cos\alpha}\right) \\ &= U\,(AD - BD) \quad \text{(Fig. 3)}. \end{aligned}$$

This agrees with Clerk Maxwell's result, *Electricity and Magnetism*, i., § 225.

11. The foregoing proofs cannot be considered satisfactory when the two planes, and therefore the two spheres obtained by inversion, cut at an angle greater than two right angles. There is, however, nothing in the expressions obtained which should lead us to look for discontinuity on passing this angle; moreover, the verification in § 7 shews that even in this case our expressions are accurate. Accordingly, in what follows, we shall assume the same expressions to hold for all inclinations. It will be seen that the assumption is borne out by several of the following verifications.

12. VERIFICATIONS.

(*a.*) *Sphere freely charged.*—If we write $2\alpha = 2\beta = \gamma = \pi$, the conductor becomes one sphere. If we put $t = s^2$, the charge E_a on one

half the conductor becomes on reduction

$$\frac{Ua}{\pi}\,.\,2\int_0^1 \frac{ds}{1+s^2}$$
$$= \tfrac{1}{2}\,Ua.$$

(β.) *Two spheres in contact.*—Let γ, and therefore α, β, become infinitely small; the conductor is then made up of two spheres in contact. The second and fourth integrals in E_a vanish, since $t^{\frac{2\pi}{\gamma}}$ becomes infinitely small everywhere between the limits of integration.

Since $$\alpha : \beta : \gamma = b : a : a+b,$$

we find the well-known result given by Poisson, viz.,

$$E_a = U\,\frac{ab}{a+b}\int_0^1 \frac{t^{\frac{b}{a+b}-1}}{1-t}\,dt.$$

(γ.) *Circular lens of infinitely fine thickness.*—Here we write $\alpha = \pi - \alpha'$, $\beta = \pi - \beta'$, $\gamma = 2\pi - \alpha' - \beta'$, and make α' and β' infinitely small, in which case, as before,

$$\alpha' : \beta' : \alpha' + \beta' = b : a : a+b.$$

Let ρ be the radius of the lens

$$E_a = \frac{U\rho}{\pi}\int_0^1 \frac{t^{-\frac{1}{2}}}{(1+t^{\frac{1}{2}})^2}\,dt$$
$$= \frac{U\rho}{\pi}.$$

The capacity, as is well known, is $\dfrac{2}{\pi}\,\rho$.

8. *A spherical bowl.*—We write $\gamma = 2\pi$, and obtain

$$E_a - E_b = \tfrac{1}{2}\,Ua\,(1+\cos\alpha).$$

This result is in agreement with Sir W. Thomson's beautiful solution of a bowl freely charged. He has shewn that the difference between the outside and inside electrification is a surface distribution on the outside of density $U \div 4\pi a$.

13. (a.) *Capacity of a spherical bowl.*—If α be the angular radius of the opening, a the radius of the bowl, then

$$E_a = \frac{Ua\sin\alpha}{2\pi}\left\{\begin{array}{l} 2\displaystyle\int_0^1 \frac{t^{\frac{\alpha}{2\pi}-1} + t^{-\frac{\alpha}{2\pi}+1} - 2}{(1-t)^2}\,dt \\[2ex] -\displaystyle\int_0^1 \frac{t^{\frac{\alpha}{2\pi}-1} - 1 + t^{-\frac{1}{2}}\left(t^{1-\frac{\alpha}{2\pi}} - 1\right)}{(1-t)\,(1-t^{\frac{1}{2}})}\,dt. \end{array}\right.$$

These integrals do not, in general, seem capable of being exhibited in finite terms. In the case of $a = \frac{1}{2}\pi$, however, we find

$$E_a = \frac{\pi+1}{2\pi}\, Ua.$$

Hence
$$E_b = E_a - \tfrac{1}{2}Ua$$
$$= \frac{1}{2\pi}\, Ua;$$

therefore
$$E_a + E_b = \frac{\pi+2}{2\pi}\, Ua.$$

The capacity of a hemispherical bowl is therefore

$$\left(\frac{1}{2} + \frac{1}{\pi}\right) a.$$

(β.) *Capacity of a conductor in the form of a hemisphere and its base.*

If we put $\beta = \pi$, we come upon the case of a lens-shaped figure one face of which is flat. An examination of the integrals in E_a will shew that they can be evaluated for various values of γ which will suggest themselves. Let us take only the case when $\gamma = \frac{3}{2}\pi$. We shall then

find
$$E_a = \left(1 - \frac{1}{\pi}\right) Ua,$$

and
$$E_b = E_a - \tfrac{1}{2}\, Ua.$$

The capacity is
$$\left(\frac{3}{2} - \frac{2}{\pi}\right) a.$$

(γ.) *Coulomb's Proof Plane.*—We here suppose that the two spheres cut at an angle nearly equal to π. We thus find

$$E_a - E_b = Ua \cos a,$$

where a is a small angle. We may also take, as a first approximation,

$$E_a + E_b = Ua.$$

Hence
$$\frac{E_b}{E_a + E_b} = \frac{a^2}{4} = \frac{\pi a^2 a^2}{4\pi a^2} = \frac{\text{Area of the proof plane}}{\text{Area of the conductor}}.$$

If we wish to proceed to a closer approximation, we must use the expression for E_b as found in § 9. Let the surfaces cut at an angle $\pi - \gamma'$, where γ' is a small angle. Then, after expansion of exponentials and reduction, we find

$$E_b = Ua \frac{\sin a}{\pi - \gamma'} \left\{ -2\left(\frac{a}{\pi} + \frac{a\gamma'}{\pi^2}\right) \int_0^1 \frac{\log t}{1-t^2} dt - \frac{a\gamma'}{\pi^2} \int_0^1 \frac{(\log t)^2}{(1+t)^2} dt \right\}$$
$$= Ua \frac{a}{\pi - \gamma'} \left\{ 2\left(\frac{a}{\pi} + \frac{a\gamma'}{\pi^2}\right) \frac{\pi^2}{8} - \frac{a\gamma'}{\pi^2} \frac{\pi^2}{6} \right\}$$
$$= Ua \frac{a^2}{4} + Ua \frac{a^2\gamma'}{3\pi}.$$

The second term is a correction upon the former result. The proportion of charge carried off by the proof plane is therefore, to the second approximation,

$$\frac{\text{Area of the plane}}{\text{Area of the conductor}} \times \left(1+\frac{4\gamma'}{3\pi}\right).$$

January 13*th,* 1881.

S. ROBERTS, Esq., F.R.S., President, in the Chair.

Miss C. A. Scott, Emmanuel House, Cambridge; Messrs. James Parker Smith, M.A., Fellow of Trinity College, Cambridge; Oscar Howard Mitchell, Fellow of Johns Hopkins University, Baltimore; and Thomas Craig, United States Coast Survey Office, Washington, were elected members.

The Chairman, with a few introductory remarks, called upon Dr. Hirst to speak upon the loss the Society had sustained by the death of its first Foreign Member, M. Chasles. Dr. Hirst rapidly glanced at the career and work of the deceased geometer, and stated some interesting particulars in illustration of the simplicity of character, geniality, and kind-heartedness of his old tutor and friend.

On the suggestion of the Chairman, a vote of thanks was passed to Mr. Howard Elphinstone for his present of several valuable mathematical works to the Library.

The following communications were made:—

"On an apparently Paradoxical Relation of the Circle, Parabola, and Hyperbola:" Mr. A. J. Ellis, F.R.S.

"A Proof of the Differential Equation which is satisfied by the Hyper-geometric Series:" Rev. T. R. Terry, M.A.

"On the Periodicity of Hyper-elliptic Integrals of the First Class:" Mr. W. R. Westropp Roberts, M.A.

"On the Tangents drawn from a Point to a Nodal Cubic:" Mr. R. A. Roberts, B.A.

"Sur une Propriété du Paramètre de la Transformée Canonique des Formes Cubiques Ternaires:" Signor Brioschi (Milan).

"Note on a Kinematical Theorem connected with the Rectilinear Courses of two Uniformly Sailing Vessels:" Mr. C. W. Merrifield, F.R.S.

"A Partition-problem connected with a Triangle and its successive Pedal Triangles:" Mr. J. W. L. Glaisher, F.R.S.

The following presents were made to the library :—

"American Journal of Mathematics," vol. iii., No. 2, June, 1880.

"Atti della R. Accademia dei Lincei—Transunti," Vol. v., Fasc. 3°, 4°; Roma, 1881.

"Beiblätter zu den Annalen der Physik und Chemie," Band v., Stück 1; Leipzig, 1881.

"Études Géométriques et Cinématiques—Note sur quelques questions de Géométrie et de Cinématique, et Réponse aux réclamations de M. l'Abbé Aoust," par E. J. Habich, Lima, 1880: from the Author.

And the above referred to collection from Mr. Howard Elphinstone.

On the Periods of the First Class of Hyper-Elliptic Integrals.

By William R. Westropp Roberts, M.A.

[*Read January* 13*th*, 1881.]

I propose to investigate the periods of hyper-elliptic integrals by a method analogous to that which has been adopted by Schlömilch for the determination of the periods of elliptic integrals. By this method I determine the periods of hyper-elliptic functions without integrating the system of equations,

$$(1)\ldots\ldots\begin{cases} F(u_1)\,du_1 \;\;\;+F(u_2)\,du_2 \;\;\;+F(u_3)\,du_3=0,\\ F(u_1)\,u_1du_1+F(u_2)\,u_2du_2+F(u_3)\,du_3=0,\end{cases}$$

where $$F(u)\equiv\frac{1}{\sqrt{u(1-u)(1-k_1^2u)(1-k_2^2u)(1-k_3^2u)}}.$$

By assuming $u=z^2$, these equations are transformed into the following :—

$$(2)\ldots\ldots\begin{cases} f(z_1)\,dz_1 \;\;\;+f(z_2)\,dz_2 \;\;\;+f(z_3)\,dz_3 \;\;\;=0,\\ f(z_1)\,z_1^2\,dz_1+f(z_2)\,z_2^2\,dz_2+f(z_3)\,z_3^2\,dz_3=0,\end{cases}$$

$f(z)$ being equal to

$$\frac{1}{\sqrt{(1-z^2)(1-k_1^2z^2)(1-k_2^2z^2)(1-k_3^2z^2)}}\equiv\frac{1}{\sqrt{Z}}.$$

Let us now consider the integral $\int_0^z f(z)\,dz$, in which we suppose z

to be a complex variable, and let us examine the different values of this integral for different routes of integration, and let $[f(z)]$ denote the absolute value of $f(z)$.

We shall suppose k_1, k_2, k_3 to be all real, less than unity, and arranged in ascending order of magnitude.

We see that $f(z)$ has eight singular points

$$z = \pm 1, \quad z = \pm \frac{1}{k_3}, \quad z = \pm \frac{1}{k_2}, \quad z = \pm \frac{1}{k_1},$$

situated on the axis of x, which we shall call A, A'; B, B'; C, C'; D, D'. Any two of these points, such as A and A', are then at equal distances from the origin O. Now the revolutions of z round any one of these points gives a change of sign to $f(z)$; they are consequently points of ramification (Verzweigungspunkte).

Now suppose that z describes a closed curve $OP_1P_2P_3O$ surrounding the point A, and let $I(A)$ be the corresponding value of $\int f(z)\,dz$. We shall have then, since we may substitute for the route $OP_1P_2P_3O$ the

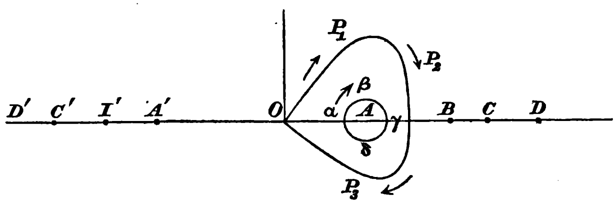


FIG. 1.

route $O\alpha\,.\,\alpha\beta\gamma\delta\alpha\,.\,\alpha O$, $\alpha\beta\gamma\delta$ being a circle of radius r whose centre is at A,
$$I(A) = I(O\alpha) + I(\alpha\beta\gamma\delta\alpha) + I(\alpha O).$$

To evaluate these integrals, we put $z = 1 - r$ in the first and third, and $z = 1 - re^{i\theta}$ in the second; then, since $f(z)$ changes sign after z describes the circle $\alpha\beta\gamma\delta$, we find

$$I(A) = \int_0^{1-r} f(z)\,dz - ir\int_0^{2\pi} f(1 - re^{i\theta})\,e^{i\theta}\,d\theta - \int_{1-r}^0 f(z)\,dz.$$

The integral relative to the circle is easily seen to vanish when r is zero, so that we find

$$I(A) = \int_0^1 f(z)\,dz - \int_1^0 f(z)\,dz = 2\int_0^1 f(z)\,dz.$$

The function $f(z)$ has now returned to O with a negative sign, so

that, for a second revolution round A, the value of the integral will be $-2\int_0^1 f(z)\,dz$. If, with Schlömilch, we denote by $I_n(A)$, $I_n(A')$ the values of the integral after n revolutions round A and A' respectively, $f(z)$ being supposed positive at the commencement of the circuits, we shall have the following system of equations, as the values of the integrals round the other points of ramification follow by symmetry :—

$$(3)\ldots\ldots\begin{cases} I_{2n+1}(A) = 2\int_0^1 f(z)\,dz, & I_{2n+1}(A') = -2\int_0^1 f(z)\,dz, \\ I_{2n+1}(B) = 2\int_0^{\frac{1}{k_3}} f(z)\,dz, & I_{2n+1}(B') = -2\int_0^{\frac{1}{k_3}} f(z)\,dz, \\ I_{2n+1}(C) = 2\int_0^{\frac{1}{k_2}} f(z)\,dz, & I_{2n+1}(C') = -2\int_0^{\frac{1}{k_2}} f(z)\,dz, \\ I_{2n+1}(D) = 2\int_0^{\frac{1}{k_1}} f(z)\,dz, & I_{2n+1}(D') = -2\int_0^{\frac{1}{k_1}} f(z)\,dz, \end{cases}$$

$f(z)$ returning to O with a change of sign ;

$$(4)\ldots\ldots\begin{cases} I_{2n}(A) = I_{2n}(B) = I_{2n}(C) = I_{2n}(D) = 0, \\ I_{2n}(A') = I_{2n}(B') = I_{2n}(C') = I_{2n}(D') = 0. \end{cases}$$

If z moves on a closed curve, as in the figure, surrounding both A and A', then

$$I(A)+I(A') = 4\int_0^1 f(z)\,dz,$$

for, after z has described the curve surrounding A, $f(z)$ returns to O with, and therefore commences the circuit of A' with,

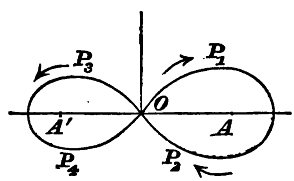


Fig. 2.

a negative sign ; $I(A')$ in this case is then $2\int_0^1 f(z)\,dz$; and, obviously, if z describes a times this curve, we must have

$$I_a(A)+I_a(A') = 4a\int_0^1 f(z)\,dz.$$

If we now consider the integral $\int_0^z f(z)\,z^2\,dz$, we see that the function $f(z)\,z^2$ has the same points of ramification as the function $f(z)$, so that, if we denote by $I'(A)$, $I'(B)$, $I'(C)$, $I'(D)$ what $I(A)$, $I(B)$, $I(C)$, and $I(D)$ become respectively when we substitute $f(z)\,z^2$ for $f(z)$, we shall have, by a process of reasoning precisely similar to that which

we employed for the determination of the latter functions,

$$(5)\ldots\ldots\begin{cases} I'_{2n+1}(A) = 2\int_0^1 f(z)\,z^2\,dz, & I'_{2n+1}(A') = -2\int_0^1 f(z)\,z^2\,dz, \\ I'_{2n+1}(B) = 2\int_0^{\frac{1}{k_2}} f(z)\,z^2\,dz, & I'_{2n+1}(B') = -2\int_0^{\frac{1}{k_2}} f(z)\,z^2\,dz, \\ I'_{2n+1}(C) = 2\int_0^{\frac{1}{k_3}} f(z)\,z^2\,dz, & I'_{2n+1}(C') = -2\int_0^{\frac{1}{k_3}} f(z)\,z^2\,dz, \\ I'_{2n+1}(D) = 2\int_0^{\frac{1}{k_1}} f(z)\,z^2\,dz, & I'_{2n+1}(D') = -2\int_0^{\frac{1}{k_1}} f(z)\,z^2\,dz, \end{cases}$$

$f(z)\,z^2$ returning to O with a change of sign;

$$(6)\ldots\ldots\begin{cases} I'_{2n}(A) = I'_{2n}(B) = I'_{2n}(C) = I'_{2n}(D) = 0, \\ I'_{2n}(A') = I'_{2n}(B') = I'_{2n}(C') = I'_{2n}(D') = 0, \end{cases}$$

$f(z)\,z^2$ returning to O with the same sign.

Also, if z moves a times round a closed curve $OP_1P_2P_3O$, as in Fig. 2, surrounding both A and A',

$$I'_a(A) + I'_a(A') = 4a\int_0^1 f(z)\,z^2dz.$$

The most general route of integration from O to a point P, representing the variable $z = x+iy$, consists of a number of circuits round the eight points of ramification, and thus of the rectilinear route from O to P. Let us first consider the effect of the points A and A' on the integral $\int_0^z f(z)\,dz$. We may suppose z to describe a times, as in Fig. 2, a curve surrounding both A and A', and then to describe β times a curve surrounding either A or A'; in the first case we shall have

$$\int_0^z f(z)\,dz = \{I_a(A) + I_a(A')\} + I_\beta(A) \pm \int_0^z [f(z)]\,dz$$

$$= 4a\int_0^1 f(z)\,dz + I_\beta(A) \pm \int_0^z [f(z)]\,dz,$$

and in the second

$$\int_0^z f(z)\,dz = 4a\int_0^1 f(z)\,dz + I_\beta(A') \pm \int_0^z [f(z)]\,dz.$$

We see, then, referring to (3) and (4), that in both cases the sign of $f(z)$ in the linear integral is positive or negative according as β is an even or odd integer. It is now easy to infer that the *general value* of the integral $\int_0^z f(z)\,dz$ is

$$(7)\ldots\ldots \int_0^s f(z)\,dz$$

$$= aI(A)+bI(B)+cI(C)+dI(D)+(-1)^{a+b+c+d}\int_0^s [f(z)]\,dz\,;$$

and, by parity of reasoning, the *general value* of the integral $\int_0^s f(z)\,z^2dz$ is

$$(8)\ldots\ldots \int f(z)\,z^2\,dz$$

$$= aI'(A)+bI'(B)+cI'(C)+dI'(D)+(-1)^{a+b+c+d}\int_0^s [f(z)\,z^2]\,dz,$$

a, b, c, and d being positive or negative integers.

Since $\quad F(u) \equiv u\,(1-u)\,(1-k_1^2\,u)\,(1-k_2^2\,u)\,(1-k_3^2\,u),$

it follows that for all values of u increasing from

$$\left\{\begin{matrix} -\infty & \text{to} & 0 \\ 1 & \text{to} & \dfrac{1}{k_3^2} \\ \dfrac{1}{k_2^2} & \text{to} & \dfrac{1}{k_1^2} \end{matrix}\right\}$$

$F(u)$ is negative; while, for values of u increasing from

$$\left\{\begin{matrix} 0 & \text{to} & 1 \\ \dfrac{1}{k_3^2} & \text{to} & \dfrac{1}{k_2^2} \\ \dfrac{1}{k_1^2} & \text{to} & \infty \end{matrix}\right\}$$

$F(u)$ is positive. Let us call the integral

$$\int_{u_2}^{u_1} F(u)\,du \equiv I\binom{u_1}{u_2},\quad \int_{u_2}^{u_1} F(u)\,u\,du \equiv I'\binom{u_1}{u_2},$$

the upper letter in $I\binom{u_1}{u_2}$ or $I'\binom{u_1}{u_2}$ corresponding to the superior limit of the integral, and the lower to the inferior. We have then

$$(9)\ldots\ldots\left\{\begin{aligned} I(A) &= I\binom{1}{0}, \\ I(B) &= I\binom{1}{0}-I\binom{\frac{1}{k_3^2}}{1}, \\ I(C) &= I\binom{1}{0}-I\binom{\frac{1}{k_3^2}}{1}+I\binom{\frac{1}{k_2^2}}{\frac{1}{k_3^2}}, \\ I(D) &= I\binom{1}{0}-I\binom{\frac{1}{k_3^2}}{1}+I\binom{\frac{1}{k_2^2}}{\frac{1}{k_3^2}}-I\binom{\frac{1}{k_1^2}}{\frac{1}{k_2^2}}, \end{aligned}\right.$$

and a similar set of equations gives the values of $I'(A)$, $I'(B)$, &c. Now, Jacobi has shown that the following relations exist betwen the I's:

$$
(10)\ldots\ldots \begin{cases}
1^\circ.\ I\binom{0}{-\infty} - I\binom{\frac{1}{k_3^2}}{1} + I\binom{\frac{1}{k_1^2}}{\frac{1}{k_2^2}} = 0, \\[2ex]
2^\circ.\ I\binom{1}{0} - I\binom{\frac{1}{k_2^2}}{\frac{1}{k_3^2}} + I\binom{\infty}{\frac{1}{k_1^2}} = 0, \\[2ex]
3^\circ.\ I'\binom{0}{-\infty} - I'\binom{\frac{1}{k_3^2}}{1} + I'\binom{\frac{1}{k_1^2}}{\frac{1}{k_2^2}} = 0, \\[2ex]
4^\circ.\ I'\binom{1}{0} - I'\binom{\frac{1}{k_2^2}}{\frac{1}{k_3^2}} + I'\binom{\infty}{\frac{1}{k_1^2}} = 0.
\end{cases}
$$

The integrals which occur in equations 1° and 3° are imaginary, and those occurring in equations 2° and 4° are real. A reduction of these quantities to the normal form of hyper-elliptic integrals, viz.,

$$\int_0^{\frac{1}{2}\pi} \frac{(\alpha+\beta\sin^2\theta)\,d\theta}{\sqrt{(1-l^2\sin^2\theta)(1-m^2\sin^2\theta)(1-n^2\sin^2\theta)}},$$

will be found in Mr. M. Roberts' "Tract on the Addition of Elliptic and Hyper-elliptic Integrals."

The general values of the integrals $\int_0^z f(z)\,dz$ and $\int_0^z f(z)\,z^2\,dz$ are now easily seen to be

$$
(11)\ldots\ldots \begin{cases}
\int_0^z f(z)\,dz = I + (-1)^{l+n}\int_0^z [f(z)]\,dz, \\[1ex]
\int_0^z f(z)\,z^2\,dz^2 = I' + (-1)^{l+n}\int_0^z [f(z)\,z^2]\,dz,
\end{cases}
$$

where $I \equiv lI\binom{1}{0} + mI\binom{\frac{1}{k_2^2}}{\frac{1}{k_3^2}} + nI\binom{\infty}{\frac{1}{k_1^2}} + \lambda I\binom{\frac{1}{k_3^2}}{1} + \mu I\binom{\frac{1}{k_1^2}}{\frac{1}{k_2^2}} + \nu I\binom{0}{-\infty}$,

$$I' \equiv lI\binom{1}{0} + mI\binom{\frac{1}{k_2^2}}{\frac{1}{k_3^2}} + nI\binom{\infty}{\frac{1}{k_1^2}} + \lambda I\binom{\frac{1}{k_3^2}}{1} + \mu I\binom{\frac{1}{k_1^2}}{\frac{1}{k_2^2}} + \nu I\binom{0}{-\infty}.$$

Let $\qquad \int_0^z f(z)\,dz = L(z), \quad \int_0^z f(z)\,z^2\,dz = M(z).$

We see, then, that $\qquad z = (-1)^{l+n}\,z'$

satisfies *both* the transcendental equations

$$
(12)\ldots\ldots \begin{cases}
L(z) = I + L(z'), \\
M(z) = I' + M(z').
\end{cases}
$$

Let us now assume $z^2+\zeta^2=1$, and put

$$\chi(\zeta)=\frac{1}{\sqrt{(1-\zeta^2)\,(h_1^2+k_1^2\zeta^2)\,(h_2^2+k_2^2\zeta^2)\,(h_3^2+k_3^2\zeta^2)}},$$

where $$h_1^2+k_1^2=1,\quad h_2^2+k_2^2=1,\quad h_3^2+k_3^2=1;$$

we find $$\int_0^z f(z)\,dz=\int_\zeta^1 \chi(\zeta)\,d(\zeta).$$

Now the function $\chi(\zeta)$ has eight points of ramification,

$$\zeta=\pm 1,\quad \zeta=\pm\frac{ih_3}{k_3},\quad \zeta=\pm\frac{ih_2}{k_2},\quad \zeta=\pm\frac{ih_1}{k_1},$$

which we shall call A_1, A_1'; B_1, B_1'; C_1, C_1'; D_1, D_1'. We find, then,

$$(13)\ldots\ldots\left\{\begin{aligned}
I(A_1)&=2\int_0^1\chi(\zeta)\,d\zeta=2\int_0^1 f(z)\,dz=I\binom{1}{0},\\
I(B_1)&=2\int_0^{\frac{ih_3}{k_3}}\chi(\zeta)\,d\zeta=2\int_{\frac{1}{k_3}}^1 f(z)\,dz=-I\binom{\frac{1}{k_3^2}}{1},\\
I(C_1)&=2\int_0^{\frac{ih_2}{k_2}}\chi(\zeta)\,d\zeta=2\int_{\frac{1}{k_2}}^1 f(z)\,dz=-I\binom{\frac{1}{k_3^2}}{1}-I\binom{\frac{1}{k_2^2}}{\frac{1}{k_3^2}},\\
I(D_1)&=2\int_0^{\frac{ih_1}{k_1}}\chi(\zeta)\,d\zeta=2\int_{\frac{1}{k_1}}^1 f(z)\,dz=-I\binom{\frac{1}{k_3^2}}{1}-I\binom{\frac{1}{k_2^2}}{\frac{1}{k_3^2}}-I\binom{\frac{1}{k_1^2}}{\frac{1}{k_2^2}}.
\end{aligned}\right.$$

The general values of the integrals $\int_0^\zeta \chi(\zeta)\,d\zeta$ and $\int_0^\zeta (1-\zeta^2)\,\chi(\zeta)\,d\zeta$ are then easily seen to be

$$(14)\ldots\ldots\left\{\begin{aligned}
\int_0^\zeta \chi(\zeta)\qquad\qquad &=I+(-1)^{l+\lambda+n+\nu}\int_0^\zeta[\chi(\zeta)]\,d\zeta,\\
\int_0^\zeta \chi(1-\zeta^2)\,d\zeta&=I'+(-1)^{l+\lambda+n+\nu}\int_0^\zeta[\chi(\zeta)(1-\zeta^2)]\,d\zeta.
\end{aligned}\right.$$

Hence, if we put $\zeta=\sqrt{1-z^2}=D(z)$, we see that

$$D(z)=(-1)^{l+\lambda+n+\nu}D(z')$$

satisfies *both* the transcendental equations

$$\int_0^{\zeta'}\chi(\zeta)\,d\zeta\qquad\qquad=I+\int_0^\zeta\chi(\zeta)\,d\zeta,$$

$$\int_0^{\zeta'}\chi(\zeta)\,(1-\zeta^2)\,d\zeta=I'+\int_0^\zeta\chi(\zeta)\,(1-\zeta^2)\,d\zeta,$$

or both the equations (12).

Let now $D(k_1 z)=\sqrt{1-k_1^2 z^2}$, $D(k_2 z)=\sqrt{1-k_2^2 z^2}$, $D(k_3 z)=\sqrt{1-k_3^2 z^2}$; then, by a similar treatment of the functions $D(k_1 z)$, $D(k_2 z)$, and $D(k_3 z)$, I arrive at a number of like conclusions, which are found in the following scheme, each equation of which is agreeable to *both* the transcendental equations (12),

$$(15)\ldots\ldots \begin{cases} 1^\circ. & z=(-1)^{l+n} z', \\ 2^\circ. & D(z)=(-1)^{l+n+\lambda+\nu} D(z'), \\ 3^\circ. & D(k_3 z)=(-1)^{\lambda+\nu+m+n} D(k_3 z'), \\ 4^\circ. & D(k_2 z)=(-1)^{m+n+\mu+\nu} D(k_2 z'), \\ 5^\circ. & D(k_1 z)=(-1)^{\mu+\nu} D(k_1 z'). \end{cases}$$

I now consider the transcendental system

$$(16)\ldots\ldots \begin{cases} L(z_1)+L(z_2)+I = L(z_1')+L(z_2'), \\ M(z_1)+M(z_2)+I' = M(z_1')+M(z_2'), \end{cases}$$

which results from putting $L(z_3)=I$ and $M(z_3)=I'$ in the system

$$(17)\ldots\ldots \begin{cases} L(z_1)+L(z_2)+L(z_3) = L(z_1')+L(z_2'), \\ M(z_1)+M(z_2)+M(z_3) = M(z_1')+M(z_2'). \end{cases}$$

If, in the latter system, we consider z_3 as a constant, there are four values of z_1^2 and z_2^2 which satisfy its algebraic integrals; for, since there is an ambiguity in the sign affecting the radical in $L(z)$, if we suppose one and the same radical always affected with the same sign, there are as many algebraic solutions as there are modes of interpreting the equations

$$+\int_0^{z_1}\frac{dz_1}{\sqrt{z_1}} \pm \int_0^{z_2}\frac{dz}{\sqrt{z}} \pm \int_0^{z_3}\frac{dz}{\sqrt{z}} = \text{a constant},$$

$$+\int_0^{z_1}\frac{z^2dz}{\sqrt{z}} \pm \int_0^{z_2}\frac{z^2dz}{\sqrt{z}} \pm \int_0^{z_3}\frac{z^2dz}{\sqrt{z}} = \text{a constant},$$

and these are readily seen to be four.

The particular case under consideration corresponds to that in which $z_3=0$, and there are then but two values of z_1^2 and z_2^2. The following system of equations is now easily seen to be the algebraic equivalent of the transcendental system (16),

$$(18)\ldots\ldots \begin{cases} 1^\circ. & z_1 z_2 = (-1)^{l+n} z_1' z_2', \\ 2^\circ. & D(z_1)\,D(z_2) = (-1)^{l+n+\lambda+\nu} D(z_1')\,D(z_2'), \\ 3^\circ. & D(k_3 z_1)\,D(k_3 z_2) = (-1)^{\lambda+\nu+m+n} D(k_3 z_1')\,D(k_3 z_2'), \\ 4^\circ. & D(k_2 z_1)\,D(k_2 z_2) = (-1)^{m+n+\mu+\nu} D(k_2 z_1')\,D(k_2 z_2'), \\ 5^\circ. & D(k_1 z_1)\,D(k_1 z_2) = (-1)^{\mu+\nu} D(k_1 z_1')\,D(k_1 z_2')\,; \end{cases}$$

and since this system gives two values for z_1^2 and z_2^2, there is no other algebraic integral.

If we now put $s = \sin\phi$, and let

$$\int_0^{\phi_1} \frac{d\phi}{\Delta(\phi)} + \int_0^{\phi_2} \frac{d\phi}{\Delta(\phi)} = \Phi,$$

$$\int_0^{\phi_1} \frac{\sin^2\phi\, d\phi}{\Delta(\phi)} + \int_0^{\phi_2} \frac{\sin^2\phi\, d\phi}{\Delta(\phi)} = \Phi',$$

we may consider the several products $\sin\phi_1 \sin\phi_2$, $\cos\phi_1 \cos\phi_2$, &c., as functions of Φ and Φ'; and let us represent this by the notation

$$\sin\phi_1 \sin\phi_2\, (\Phi, \Phi').$$

We find, then, in conclusion

1°. $\sin\phi_1 \sin\phi_2\, (\Phi + I, \Phi' + I') = (-1)^{l+n} \sin\phi_1 \sin\phi_2\, (\Phi, \Phi')$,

2°. $\cos\phi_1 \cos\phi_2\, (\Phi + I, \Phi' + I') = (-1)^{l+n+\lambda+\nu} \cos\phi_1 \cos\phi_2\, (\Phi, \Phi')$,

3°. $\sqrt{1-k_3^2\sin^2\phi_1}\,\sqrt{1-k_3^2\sin^2\phi_2}\,(\Phi+I, \Phi'+I')$
$\qquad = (-1)^{\lambda+\nu+m+n}\sqrt{1-k_3^2\sin^2\phi_1}\,\sqrt{1-k_3^2\sin^2\phi_2}\,(\Phi, \Phi')$,

4°. $\sqrt{1-k_2^2\sin^2\phi_1}\,\sqrt{1-k_2^2\sin^2\phi_2}\,(\Phi+I, \Phi'+I')$
$\qquad = (-1)^{m+n+\mu+\nu}\sqrt{1-k_2^2\sin^2\phi_1}\,\sqrt{1-k_2^2\sin^2\phi_2}\,(\Phi, \Phi')$,

5°. $\sqrt{1-k_1^2\sin^2\phi_1}\,\sqrt{1-k_1^2\sin^2\phi_2}\,(\Phi+I, \Phi'+I')$
$\qquad = (-1)^{\mu+\nu}\sqrt{1-k_1^2\sin^2\phi_1}\,\sqrt{1-k_1^2\sin^2\phi_2}\,(\Phi, \Phi')$,

where $I = lI\begin{pmatrix}1\\0\end{pmatrix} + mI\begin{pmatrix}\frac{1}{k_2^2}\\ \frac{1}{k_3^2}\end{pmatrix} + nI\begin{pmatrix}\infty\\ \frac{1}{k_1^2}\end{pmatrix} + \lambda I\begin{pmatrix}\frac{1}{k_3^2}\\ 1\end{pmatrix} + \mu I\begin{pmatrix}\frac{1}{k_1^2}\\ \frac{1}{k_2^2}\end{pmatrix} + \nu I\begin{pmatrix}0\\ -\infty\end{pmatrix}$,

$$I' = lI'\begin{pmatrix}1\\0\end{pmatrix} + mI'\begin{pmatrix}\frac{1}{k_2^2}\\ \frac{1}{k_3^2}\end{pmatrix} + nI'\begin{pmatrix}\infty\\ \frac{1}{k_1^2}\end{pmatrix} + \lambda I'\begin{pmatrix}\frac{1}{k_3^2}\\ 1\end{pmatrix} + \mu I'\begin{pmatrix}\frac{1}{k_1^2}\\ \frac{1}{k_2^2}\end{pmatrix} + \nu I'\begin{pmatrix}0\\ -\infty\end{pmatrix},$$

and $$I\begin{pmatrix}u_1\\u_2\end{pmatrix} = \int_{u_2}^{u_1} \frac{du}{\sqrt{u(1-u)(1-k_1^2u)(1-k_2^2u)(1-k_3^2u)}},$$

$$I'\begin{pmatrix}u_1\\u_2\end{pmatrix} = \int_{u_1}^{u_1} \frac{u\,du}{\sqrt{u(1-u)(1-k_1^2u)(1-k_2^2u)(1-k_3^2u)}}.$$

The functions I have discussed, viz., $\sin\phi_1 \sin\phi_2\, (\Phi, \Phi')$, &c., are then periodic functions of two variables Φ and Φ', the nature of the periodicity being indicated by the system (19).

Thursday, Feb. 10*th,* 1881.

S. ROBERTS, Esq., F.R.S., President, in the Chair.

Mr. W. Woodruff Beman, University of Michigan, was elected a member, and Messrs. R. C. Rowe and J. Parker Smith were admitted into the Society.

The following communications were made:—

"On some Integrals expressible in terms of the First Complete Elliptic Integral and of Gamma-Functions," Mr. J. W. L. Glaisher, F.R.S.

"Some Theorems of Kinematics on a Sphere," Mr. E. B. Elliott, M.A.

"Supplement on Binomial Biordinals," Sir James Cockle, F.R.S.

"An Application of Conjugate Functions," Mr. E. J. Routh, F.R.S.

"Note on Abel's Theorem," Mr. T. Craig.

The following presents were received:—

"Annali di Matematica," Serie ii^a, Tomo x., fasc. 2° (Gennajo, 1881); Milano, 1881.

"Educational Times," Feb. 1881.

"Physical Society of London—Proceedings," Vol. iv., Pt. i., August–December, 1880.

"A Memoir on the Single and Double Theta-Functions (from the "Phil. Trans.," Pt. iii., 1880): from Prof. Cayley, F.R.S.

"Table of $\Delta^m 0^n \div \Pi(m)$ up to $m = n = 20$ (from the Cambridge "Phil. Trans.," Vol. xiii., Pt. i.): from the Author, Prof. Cayley, F.R.S.

"Mathematical Reprint from the Educational Times," Vol. xxxiv.

"Proceedings of Royal Society," Vol. xxxi., Nos. 208, 209.

"Atti della R. Accademia dei Lincei—Transunti," Vol. v., Fasc. 5°, 6°; Roma, 1881.

"Bulletin des Sciences Mathématiques et Astronomiques," 2^me Série, Tome iv.; Août 1880.

"Annales des Ponts et Chaussées—Mémoires et documents relatifs à l'art des constructions et au service de l'ingénieur, lois, décrets, arrêtés et autres actes concernant l'administration des ponts et chaussées," 1881 Janvier; Paris.

"Beiblätter zu den Annalen der Physik und Chemie," Band v., Stück 2; Leipzig, 1881.

"Annales Scientifiques de l'École Normale Supérieure," 2^e Série, Tome dixième; Année 1881; No. 2, Février; Paris, 1881.

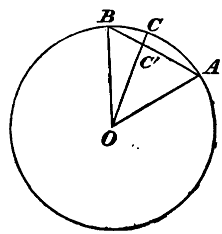
"University of Cincinnati—Publications of the Cincinnati Observatory, Micrometrical Measurements of Double Stars," 1878–1879.

Some Theorems of Kinematics on a Sphere. *By* E. B. Elliott.

[*Read Feb.* 10*th*, 1881.]

1. The theorems in question have to do with the spherical areas passed round by points of a spherical figure as it moves upon its sphere without changing size or form through a complete cycle of positions ending with its original one, and are the analogues of the theorem of plane kinematics, known as Holditch's, and of others given by Messrs. Leudesdorf and Kempe, in the "Messenger of Mathematics" for 1877 and 1878.

Let the radius of our sphere be R, and consider AB an arc of great circle upon it of given length, and consequently subtending a given angle, $\alpha+\beta$ say, at the sphere's centre. Let this arc be divided at C into two constant parts α and β. The chord joining AB is, of course, also of constant length $2R\sin\frac{1}{2}(\alpha+\beta)$, and, if the radius to C meet it in C', then $AC' : C'B = \sin\alpha : \sin\beta$, a constant ratio. Thus the consideration of the kinematics of a constant arc moving on a sphere, and a point always dividing it into two constant parts α, β, resolves itself into that of a rod of constant length, moving with its two ends on a sphere, and the point C' which always divides it in a constant ratio $\sin\alpha : \sin\beta$.



Again, since $\dfrac{OC'}{OA} = \dfrac{\sin C'AO}{\sin AC'O} = \dfrac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}$, the concentric sphere on which C' moves is of radius $R\dfrac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}$.

2. Now the rod AB, in any position, has three distinct motions open to it,—(1) a rotation about the centre of the sphere in the great circle plane which contains it; (2) a rotation about its own middle point in the plane, which contains it, and is at right angles to this one; and (3) a translation at right angles to itself in this second plane. Let it be given an infinitesimal displacement $d\theta$ of the first kind; then the two ends A, B are displaced on the sphere in virtue of it, through distances each $Rd\theta$, and the point C' through a distance $R\dfrac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}d\theta$. Again, let the rod be given simultaneously or successively displacements $d\phi$, ds of the second and third kinds respectively; then, writing $2c$ instead of $2R\sin\frac{1}{2}(\alpha+\beta)$, the length, the consequent displacements of A and B on the sphere will be $ds+cd\phi$ and $ds-cd\phi$ respectively, in directions at right angles to the first displacements of those points, and that of C' upon its own sphere will be $ds+\dfrac{\sin\beta-\sin\alpha}{\sin\beta+\sin\alpha}cd\phi$ at right

angles to its first displacement. Thus, $d(A)$ and $d(B)$ being rectangular elements of area on the given sphere contained by these rectangular displacements of A and B respectively,

$$d(A) = Rd\theta\,(ds + cd\phi)$$

and

$$d(B) = Rd\theta\,(ds - cd\phi),$$

and there is a corresponding rectangular element on the sphere which is the locus of C', viz.,

$$d(C') = R\frac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}d\theta\left\{ds - \frac{\sin\alpha - \sin\beta}{\sin\alpha+\sin\beta}cd\phi\right\}.$$

From these, eliminating $d\theta\, ds$ and $d\theta\, d\phi$, we get

$$(\sin\alpha + \sin\beta)\frac{\cos\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha+\beta)}\,d(C') = \sin\beta\, d(A) + \sin\alpha\, d(B),$$

i.e.,

$$d(C') = \frac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}\cdot\frac{\sin\beta\, d(A) + \sin\alpha\, d(B)}{\sin\alpha+\sin\beta}.$$

Now, suppose that the one end A of our rod pass just completely round the perimeter of an area (A) on the sphere, and that meanwhile the other B pass also just entirely round the perimeter of another area (B), so that C' also passes round the perimeter of an area (C') on its own sphere. Then, if the two areas (A), (B) are such that they can just completely be covered by pairs of corresponding points, each pair of which are possible simultaneous positions of the two ends of our rod, the areas can be split up entirely into elements $d(A)$, $d(B)$, $d(C')$, to which the above result applies. Summing, then, we have a like relation between the whole areas

$$(C') = \frac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}\cdot\frac{\sin\alpha\,(B) + \sin\beta\,(A)}{\sin\alpha+\sin\beta} \quad\ldots\ldots\ldots\ldots\ldots (1).$$

But now each point C' is, as has been seen, the central projection of the point C which divides the arc AB into two parts α, β; and each element $d(C)$ surrounded by C is simply the corresponding $d(C')$ multiplied by the square of the ratio of the radii of the two spheres on which they lie. Consequently, we see that, if the two ends of a moving arc of great circle of given angular length $\alpha+\beta$ pass simultaneously on the sphere just all round closed areas (A), (B), which can be covered by corresponding points as above described, the point C which divides it into two constant arcs α, β, will in the same motion pass just all round an area (C) given by

$$(C) = \frac{\cos^2\frac{1}{2}(\alpha-\beta)}{\cos^2\frac{1}{2}(\alpha+\beta)}\cdot\frac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}\cdot\frac{\sin\alpha\,(B)+\sin\beta\,(A)}{\sin\alpha+\sin\beta}$$

$$= \frac{\sin\alpha\,(B)+\sin\beta\,(A)}{\sin(\alpha+\beta)} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

In verification, proceeding to the limit when the sphere becomes a plane, this agrees with the generalised form of Holditch's theorem,

$$(C) = \frac{m(B)+n(A)}{m+n} - \frac{mn}{(m+n)^2}S,$$

for it is a case where the relative area S vanishes.

4. Now it is not always that the two spherical areas (A), (B), passed round by the two ends of an arc that moves through a complete cycle of positions to its old one again, can be entirely covered by pairs of points, each of which is distant from its conjugate by an arc equal to the moving one. It always is possible when each closed curve lies entirely without the other, in which case the moving arc returns to its first position without having made a complete rotation; but when, as is the case if one area (A) lies within the other (B), the arc has to make a complete rotation before returning to its old place (or rather, considering the arc rigidly fixed to a closely fitting spherical surface which slips on the fixed sphere as it moves, when in the motion this spherical surface returns to its first position, having in the mean while completely turned through 2π about some one of its diameters), it cannot be done. The relation (2) then has not been proved for such areas.

As preliminary to discovering the relation which takes its place, let us consider first a special motion. Let one end A of the moving arc remain fixed, so that the other B describes a circle of angular radius $\alpha+\beta$ with it as centre, and so C one of radius α. Call the spherical areas of these circles (b) and (c); then

$$(b) = 4\pi R^2 \sin^2 \tfrac{1}{2}(\alpha+\beta), \quad (c) = 4\pi R^2 \sin^2 \tfrac{1}{2}\alpha \text{ .... } (3, 4).$$

Now, generally, taking our given surrounded areas (A), (B), call (A) the inner; and within it let there be described an infinite succession of continually smaller curves, each differing infinitesimally from the preceding, and the last one being a point. Then there will be a corresponding infinite succession of curves, the first differing infinitesimally from (B), and each infinitesimally from the preceding, such that arcs of length $\alpha+\beta$ can move all round with one end on any one of them, and the other upon the corresponding one of the other system; and the last of these must be a circle (b) about the point which is the last of the first system as centre. Each of the second system of curves will lie mostly, as a rule altogether, within the preceding, and consequently the last of them, the circle (b), will generally lie entirely within the first one (B). But the method does not fail in case part of the circle lie without it, provided the natural convention be made, as has already been done implicitly, that areas passed round by a point in one sense of rotation being considered positive, those passed round in the other sense are negative.

So, also, corresponding to each curve of the A-system, and the conjugate one of the B-system, there will be a curve of a C-system; and the last of these, corresponding to the point (a) and the circle (b), will be a circle (c) of angular radius α.

Now the areas (A), $(B)-(b)$, $(C)-(c)$ are made up of elements such as those which composed the (A), (B), (C) of equation (2). We have, therefore, $(C)-(c) = \dfrac{\sin\alpha\,\{(B)-(b)\}+\sin\beta\,(A)}{\sin(\alpha+\beta)}$.

Hence, substituting for (b) and (c) from (3, 4), we have

$$(C)-\frac{\sin\alpha\,(B)+\sin\beta\,(A)}{\sin(\alpha+\beta)} = 4\pi R^2\left\{\sin^2\frac{\alpha}{2}-\frac{\sin^2\frac{1}{2}(\alpha+\beta)\sin\alpha}{\sin(\alpha+\beta)}\right\}$$

$$= -4\pi R^2\frac{\sin\frac{1}{2}\alpha\,\sin\frac{1}{2}\beta}{\cos\frac{1}{2}(\alpha+\beta)} \quad\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

In case the slipping spherical surface, to which the moving arc is supposed rigidly attached, has to make n revolutions instead of one in the course of the complete motion, the circular areas will have to be reckoned n times instead of once. Thus the general result, of which (2) and (5) are special cases, is

$$(C) = \frac{\sin\alpha\,(B)+\sin\beta\,(A)}{\sin(\alpha+\beta)} - 4n\pi R^2\frac{\sin\frac{1}{2}\alpha\,\sin\frac{1}{2}\beta}{\cos\frac{1}{2}(\alpha+\beta)} \quad\ldots\ldots\ldots\ldots(6).$$

The area of (C'), enclosed upon its own sphere by the curve which C', dividing the chord AB in the constant ratio $\sin\alpha : \sin\beta$, describes, is of course found at once from that of (C) by multiplying by the square of the ratio of the radii. Thus

$$(C') = \frac{\cos^2\frac{1}{2}(\alpha+\beta)}{\cos^2\frac{1}{2}(\alpha-\beta)}\left\{\frac{\sin\alpha\,(B)+\sin\beta\,(A)}{\sin(\alpha+\beta)} - 4n\pi R^2\frac{\sin\frac{1}{2}\alpha\,\sin\frac{1}{2}\beta}{\cos\frac{1}{2}(\alpha+\beta)}\right\} \quad\ldots\ldots\ldots (7).$$

The limiting form which (5) or (6) takes when R is made infinite, is Holditch's theorem of plane kinematics, as extended by Woolhouse.

5. In (6) let $(B) = (A)$, *i.e.*, let the two ends A, B of our moving arc go round either the same spherical curve or two curves of equal area; then we get, reducing,

$$(C)\cos\tfrac{1}{2}(\alpha+\beta) = (A)\cos\tfrac{1}{2}(\alpha-\beta)-4n\pi R^2\sin\frac{\alpha}{2}\sin\frac{\beta}{2},$$

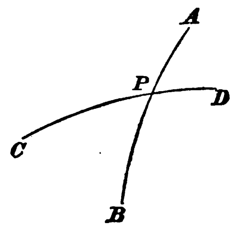
which may be written

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{(C)-(A)}{(C)+(A)-4n\pi R^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

in which n is zero and 1 respectively in the cases of the most simple and usual complete motions contemplated in (2) and (5), and is always a positive or negative integer.

6. We have already considered our moving arc AB as a part of a spherical figure, or indeed of a whole spherical surface, fitting the fixed sphere closely, and moving on it through a closed cycle of positions. Let us discuss now the areas surrounded by different points of this moving surface that are not all on the same great circle; and firstly find the locus of points which in the complete motion pass round areas equal to a given one.

Let A, B, C, D be four points of the moving surface which pass round equal areas (A), and let the arcs AB, CD meet in P. Then all the arcs in the figure are constant, and the above results apply. Using then (8), we obtain a value for $\tan\frac{AP}{2}\tan\frac{PB}{2}$ in terms of (A), (P), and constants; and, again, precisely the same value for $\tan\frac{CP}{2}\tan\frac{PD}{2}$. Thus



$$\tan\frac{AP}{2}\tan\frac{PB}{2}=\tan\frac{CP}{2}\tan\frac{PD}{2}.$$

But this is the necessary and sufficient condition that A, B, C, D lie upon the same circle, small or great, of the sphere. The locus required, then, for any given value of (A) is a circle.

Let S be the centre of this circle, ρ its radius, $\rho'=SP$; then we have

$$\tan\tfrac{1}{2}(\rho+\rho')\tan\tfrac{1}{2}(\rho-\rho')=\tan\frac{AP}{2}\tan\frac{PB}{2}=\text{const.},$$

by (8), if (P) is constant. Thus $(P)=$ const. necessitates that ρ' as well as ρ be constant. The circle, then, which is the locus of points passing round the area (P), is concentric with that giving (A). The various locus circles are therefore concentric.

Again, if (S) be the area passed round by S, we obtain at once, by applying (6) or (8) to a bisected arc through S,

$$(A)=(S)\cos\rho+4n\pi R^2\sin^2\frac{\rho}{2}\quad\dots\dots\dots\dots\dots\dots(9);$$

or, as it is perhaps more conveniently written,

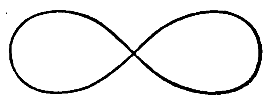
$$\cos\rho=\frac{(A)-2n\pi R^2}{(S)-2n\pi R^2}\quad\dots\dots\dots\dots\dots\dots\dots\dots(10),$$

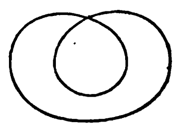
thus giving the radius of the circle which is the locus for any assigned area. We are shewn, too, that the least and greatest areas which can be passed round by any point of the sphere, are the one (S) and the other $4n\pi R^2-(S)$, which are passed round by the centre of the locus circles and the diametrically opposite point respectively. The mean

area $2n\pi R^2$ is passed round by points on the polar great circle of these singular points.

The theorem of this article is the analogue of one as to plane kinematics, given by Mr. Kempe* as the interpretation of a result by Mr. Leudesdorf. The direct analogue of Mr. Leudesdorf's theorem follows in the next.

Since obtaining all the results above (and those which follow as far as Art. 8 inclusive), my attention has been called to a comprehensive paper by M. Darboux, in the "*Bulletin des Sciences Mathématiques et Astronomiques*," for August, 1878, of which a section is devoted to obtaining the locus theorem of the present article. M. Darboux's result, however, is but special, being for the case of $n = 2$, and obtaining, instead of (10), the form $\dfrac{4\pi R^2-(S)}{\cos\rho} =$ constant, which it takes for that case. It is easy, however, to make his proof and result general. He follows the method of roulettes on the sphere, and bases his conclusion on the statement that the sum of an area on a unit sphere, and the integral change of direction in passing round its perimeter, is 2π. Now, use instead the general form of this special fact, namely, that the sum is $2k\pi$, where k is either zero or some integer, and the theorem follows as above, with perfect generality. As instances of cases where k has other values than unity, take spherical areas such as those in the two figures adjoining. In the first, where the area is the difference of the two loops, $k = 0$; and in the second, where it is their sum, $k = 2$.





7. We can readily connect the areas (A), (B), (C), (P) passed round on the sphere by four points, the position of one of which P is given with reference to the spherical triangle ABC of which the other three are vertices. Thus, let AP, BP, CP meet BC, CA, AB respectively in

* For the general case when n is not zero, Mr. Kempe's locus theorem, expressed by $(A) = (S)+n\pi\rho^2$, follows at once by proceeding to the limit with the part of the sphere near the centre of the locus circles on it. The special case of no revolution ($n = 0$) is, however, exceptional. For this case the theorem on the sphere becomes

$$(A) = (S)\cos\rho = (S)\sin p,$$

where p, the complement of ρ, is the distance from the great circle of the system of locus circles, each point of which now passes round a zero area. Proceeding then to the limit with the part of the sphere about a point of this great circle, it becomes a straight line, and the area (A), passed round by a point in what becomes the plane of motion, is seen to vary as p, the distance from that straight line. Thus Mr. Kempe's special theorem, that in the case of a non-revolutional complete motion in a plane, the loci for equal areas are straight lines instead of circles, is not, as might at first appear, inconsistent with the result above obtained on a sphere.

A', B', C'. Write down, by (6), a relation connecting (A), (A'), and (P), and another connecting (B), (C), and (A'). From these, by elimination of (A'), a result is found easily reducible to

$$(P)=\frac{\sin PA'}{\sin AA'}(A)+\frac{\sin PB'}{\sin BB'}(B)+\frac{\sin PC'}{\sin CC'}(C)$$

$$-4\pi R^2\left\{\frac{\sin\frac{1}{2}AP\sin\frac{1}{2}PA'}{\cos\frac{1}{2}AA'}+\frac{\sin AP}{\sin AA'}\cdot\frac{\sin\frac{1}{2}BA'\sin\frac{1}{2}A'C}{\cos\frac{1}{2}BC}\right\},$$

of which the part within the bracket must of course be capable of being brought by direct work to a symmetrical form.

To follow another method, which will at once obtain the result in a convenient shape, will, however, be interesting.

Take O the centre of a sphere, and let the tetrahedral coordinates of P with regard to the tetrahedron $ABCO$ be x, y, z, ω, so that, P being on the sphere, $x=\dfrac{\sin PA'}{\sin AA'}$, $y=\dfrac{\sin PB'}{\sin BB''}$, $z=\dfrac{\sin PC'}{\sin CC'}$, $\omega=1-x-y-z$. Take also S the centre of the locus circles found in the last article. We have the vector equality

$$\widehat{SP}=x\widehat{SA}+y\widehat{SB}+z\widehat{SC}+\omega\widehat{SO},$$

squaring each side of which, equating scalar parts and reducing, we get the relation in squares of straight lines

$$SP^2=xSA^2+ySB^2+zSC^2+\omega SO^2-x\omega OA^2-y\omega OB^2-z\omega OC^2$$
$$-yzBC^2-zxCA^2-xyAB^2.$$

Now, let ρ_1, ρ_2, ρ_3, ρ be the lengths of the arcs SA, SB, SC, SP; then, dividing this by R^2, we get

$$4\sin^2\frac{\rho}{2}=4\left(x\sin^2\frac{\rho_1}{2}+y\sin^2\frac{\rho_2}{2}+z\sin^2\frac{\rho_3}{2}\right)+\omega\,(1-x-y-z)$$
$$-4\left(yz\sin^2\frac{a}{2}+zx\sin^2\frac{b}{2}+xy\sin^2\frac{c}{2}\right).$$

Now, by (9), $$\sin^2\frac{\rho_1}{2}=\frac{(A)-(S)}{4n\pi R^2-2\,(S)},$$

and similarly for $\sin^2\dfrac{\rho_2}{2}$, $\sin^2\dfrac{\rho_3}{2}$, $\sin^2\dfrac{\rho}{2}$. Therefore, inserting,

$$4\{(P)-x\,(A)-y\,(B)-z\,(C)\}$$
$$=2\,(S)\left\{2\omega-\omega^2+4\left(yz\sin^2\frac{a}{2}+zx\sin^2\frac{b}{2}+xy\sin^2\frac{c}{2}\right)\right\}$$
$$+\left\{\omega^2-4\left(yz\sin^2\frac{a}{2}+zx\sin^2\frac{b}{2}+xy\sin^2\frac{c}{2}\right)\right\}4n\pi R^2.$$

But, by the equation of the sphere, the coefficient of (S) in this equation vanishes, and that of $4n\pi R^2$ reduces to 2ω. We have, there-

fore,
$$(P) = x(A)+y(B)+z(C)+\omega \,.\, 2n\pi R^2 \quad \ldots\ldots\ldots\ldots(11),$$
the relation required. It may be written in various forms symmetrical as to A, B, and C. Thus, for instance, at once
$$(P)-2n\pi R^2 = \{(A)-2n\pi R^2\}\frac{\sin PA'}{\sin AA'}+\{(B)-2n\pi R^2\}\frac{\sin PB'}{\sin BB'}$$
$$+\{(C)-2n\pi R^2\}\frac{\sin PC'}{\sin CC'} \quad \ldots\ldots\ldots\ldots(12).$$

Or, again, let r be the angular radius of the small circle circumscribing ABC, and r' the distance of P from its centre; then

ω = ratio (with proper sign) of distances of P and O from the plane ABC,
$$= -\frac{\cos r' - \cos r}{\cos r} = -\frac{2\sin\frac{1}{2}(r+r')\sin\frac{1}{2}(r-r')}{\cos r}$$
$$= -\frac{2\cos\frac{1}{2}(r+r')\cos\frac{1}{2}(r-r')}{\cos r}\tan\tfrac{1}{2}(r+r')\tan\tfrac{1}{2}(r-r')$$
$$= +\frac{\cos r+\cos r'}{\cos r}\,.\,\tan^2\frac{\tau}{2},$$
τ being the real or imaginary tangent arc from P to the circumscribing circle. Thus, inserting,
$$(P) = (A)\frac{\sin PA'}{\sin AA'}+(B)\frac{\sin PB'}{\sin BB'}+(C)\frac{\sin PC'}{\sin CC'}$$
$$+2n\pi R^2\frac{\cos r+\cos r'}{\cos r}\tan^2\frac{\tau}{2}\ldots\ldots(13);$$
which, by proceeding to the limit when R is infinite, includes Mr. Leudesdorf's theorem.

8. It is, of course, easy now to state all the above results as to areas passed round by points on the sphere as relations between the solid angles of the cones passed round by the different lines through a fixed point of a solid, which moves through a closed cycle of positions about that point; or, again, taking these lines as of finite lengths, either the same or different, between the sectorial volumes cut by these cones from the spheres which are the loci of the lines' extremities.

9. It occurs now, from facts as to these cones, to determine correlative ones as to the reciprocal cones; in other words, to pass from the points whose motion upon the sphere we have been considering, to the great circles of which they are the poles, and so determine properties of the curves which great circles of the moving spherical surface envelope, correlative to those already found for the curves which are the loci of its points.

We know that an intersection of two arcs of great circle is pole to the connector of their poles, and so that the relation between two spherical curves, of which the one is obtained as the envelope of great circles to which the points of the other are poles, is entirely reciprocal. Thus, while a point A moves, as in either of the cases above, round a spherical area (A), its polar great circle moves round and envelopes a curve whose point of contact with it in any position is pole of the tangent great circle at the corresponding position of A. We know also that the angle between two great circle arcs is equal to the angular distance between their poles. It follows that the angle between two consecutive tangents to the perimeter of (A) is equal to the angular length of the corresponding element of arc of the reciprocal envelope. Hence, summing for a complete motion, we obtain that the entire change of direction in passing all round the closed perimeter of (A) is equal to the entire angular length of the perimeter of the, of course closed, reciprocal curve which is the envelope of the polar great circle of A. Thus the actual length of the perimeter,

$$S_a = R \times \text{change of direction in passing round } (A)$$

$$= 2k\pi R - \frac{1}{R}(A) \quad\ldots\ldots\ldots\ldots(14),$$

where k equals zero, or an integer, and is the algebraic number of loops of which (A) consists, *i.e.*, the excess of the number passed round in the positive sense over that in the negative.

Hence, at once, the closed motion being such that k is constant for the poles of all great circles of the sphere that are considered, S_a is constant whenever (A) is. The result of Art. 6, then, at once reciprocates into,—"If a spherical surface closely fitting a fixed one move on it through a closed cycle of positions, the envelope on the moving surface of great circles whose envelopes on the fixed one are of constant perimeter S_a is a circle. For different values of S_a the different envelope circles are concentric. And, if S_o be the perimeter of the envelope of the great circle of which their common centre is pole, the radius ρ of any one of them is given by

$$\sin\rho = \frac{2(k-n)\pi R - S_a}{2(k-n)\pi R - S_o} \quad\ldots\ldots\ldots\ldots(15),$$

where $k-n$ is the excess of the algebraic number of loops in the curve which is the locus of the pole of the circle which envelopes S_a, over the number of times which in the complete motion the moving sphere has made complete revolutions round the axis for which that number is greatest." Great circles, in particular, which pass through the common centre of the envelope circles, envelope a constant perimeter $2(k-n)\pi R$, depending solely on k, n and the dimensions of the sphere.

In the most usual motions $k = n$, so that (15) takes the simple form $S_a = S_o \sin\rho$. Of other possible motions the most important seem to be those where $k = n+1$, to which the simple one of Art. 3 belongs ($n = 0$, $k = 1$). The equivalent of the theorem of the present article has been obtained for the case $n=2$, $k=3$, by M. Darboux in the paper already referred to.

10. Still considering k to be constant for the poles of all the great circles dealt with, we can at once write down the correlative of result (6) above, by writing in it from (14) for (A), and similarly for (B) and (C). The conclusion is, that if two great circles making an angle $\alpha+\beta$ with each other envelope curves of perimeters S_a, S_b respectively in a complete cyclic motion, then the great circle which divides the angle between them into two parts α, β, shall envelope a perimeter S_c, given by an equation that is readily reduced to

$$S_c = \frac{\sin\alpha \,.\, S_b + \sin\beta \,.\, S_a}{\sin(\alpha+\beta)} - 4(k-n)\frac{\sin\frac{1}{2}\alpha \sin\frac{1}{2}\beta}{\cos\frac{1}{2}(\alpha+\beta)} \quad \text{......(15).}$$

It is seen then that the common case for which $k = n$, which includes that contemplated in result (5) above, is, as far as lengths of envelopes go, one of exceptional simplicity, the motion giving the simplest result as to loci of points, ($k = 1$, $n = 0$) that of (2), being as to envelopes more complicated.

11. Again, we may now write down the correlative of the result of Art. 7, (11), (12), or (13). Letting ABC be the polar triangle of the one considered in that article, and letting the polar great circle of P meet its sides respectively in A', B', C', we have, that the x of that

Article $$= \frac{\sin AA'B'}{\sin AA'B} = \frac{\sin\varpi_1}{\sin p_1};$$

and, similarly, $y = \dfrac{\sin\varpi_2}{\sin p_2}$, $z = \dfrac{\sin\varpi_3}{\sin p_3}$, p_1, p_2, p_3 being the perpendicular arcs from the vertices A, B, C on the opposite sides of the triangle, and ϖ_1, ϖ_2, ϖ_3 the perpendicular arcs from those vertices upon $A'B'C'$. Thus we may state,—"If in a closed motion of a spherical figure upon its sphere the three sides of a spherical triangle upon it envelope curves of perimeters S_a. S_b, S_c respectively, the great circle on which the perpendiculars from the vertices are ϖ_1, ϖ_2, ϖ_3, shall envelope one of perimeter S_p given by

$$S_p - 2(k-n)\pi R = \{S_a - 2(k-n)\pi R\}\frac{\sin\varpi_1}{\sin p_1} + \{S_b - 2(k-n)\pi R\}\frac{\sin\varpi_2}{\sin p_2}$$
$$+ \{S_c - 2(k-n)\pi R\}\frac{\sin\varpi_3}{\sin p_3} \quad \text{......................(18),}$$

the correlative of (12), or again by

$$S_p = S_a \frac{\sin \varpi_1}{\sin p_1} + S_b \frac{\sin \varpi_2}{\sin p_2} + S_c \frac{\sin \varpi_3}{\sin p_3}$$

$$+2(k-n)\,\pi R \frac{\sin r + \sin r'}{\sin r} \tan^2 \frac{t}{2} \quad \text{........(19)},$$

where r is the angular radius of the circle inscribed in ABC, r' the angular distance of the great circle considered from the centre of that circle, and t the angle at which that great circle cuts it." The last form of result is derived from the first, just as (13) from (12).

12. The theorem of plane kinematics obtained by taking that of the last article in the limiting case when the sphere becomes a plane, is new to me. The class of cases $k=n$ alone seems to have meaning in the limit, for in a plane the points whose motion determine k are infinitely distant ones. Now, for this class of cases, the remainder on the right of (19) vanishes, so that we may enunciate,—"In any closed motion of a plane figure in its plane, if S_a, S_b, S_c be the perimeters of the curves enveloped by the sides of a plane triangle ABC in it, that enveloped by a straight line of it, on which the perpendiculars from the vertices are p, q, r respectively, shall be of perimeter S_p, given by

$$2\Delta S_p = pa\,S_a + qb\,S_b + rc\,S_c \quad \text{....................(20)},$$

where a, b, c are the sides, and Δ the area of the triangle ABC."

The plane theorems which are the limits of Arts. 9 and 10 have been previously obtained by M. Darboux by direct process in a plane, as also has been the remarkable relation

$$0 = a\,S_a + b\,S_b + c\,S_c,$$

obtained by giving p, q, r, in (20), infinite equal values.

It may perhaps be allowed me to make another remark on M. Darboux's paper in the "*Bulletin*," which has been so suggestive to me in the latter part of this one; viz., that his extension of Holditch's theorem to sectorial areas is one included in my earlier extension of it ("Messenger of Mathematics," Feb. 1878), which removed the restriction that the moving line be necessarily of constant length; and in like manner, that his final result on volumes, given as an extended analogue of one of mine in the same paper in the "Messenger," is, on the other hand, included in my theorem.

Sur une Propriété du Paramètre de la transformée Canonique des Formes cubiques ternaires. *Par* Signor Brioschi.

[*Read January* 13*th*, 1881.]

1°. Dans une lettre adressée à Mr. Hermite il y a quelques années (*Comptes Rendus*, 13 Juillet, 1863) j'ai énonçé le théorème suivant: " Soit U une forme cubique ternaire, H son hessien, S, T ses invariants; si l'on indique par x une racine quelconque de l'équation :

$$(1) \qquad x^4-6Sx^2+8Tx-3S^2=0,$$

l'équation du quatrième degré dont les racines sont les valeurs de X données par la relation :

$$(2) \qquad \sqrt{X}=\frac{1}{S}\left[aS+\tfrac{1}{2}b\,(x^3-7Sx+8T)\right]\sqrt{x}$$

est la suivante :

$$(3) \qquad X^4-6S_{ab}X^2+8T_{ab}X-3S^2_{ab}=0,$$

S_{ab}, T_{ab} étant les invariants de la forme cubique ternaire $aU+bH$, ou bien :

$$S_{ab}=Sa^4+4Ta^3b+6S^2a^2b^2+4STab^3+(4T^2-3S^3)\,b^4,$$
$$T_{ab}=Ta^6+6S^2a^5b+15STa^4b^2+20T^2a^3b^3+15S^2Ta^2b^4$$
$$+6S\,(3S^3-2T^2)\,ab^5+T\,(3S^3-8T^2)\,b^6.$$

J'observais encore dans cette même lettre que, en indiquant par x_∞, x_0, x_1, x_2 les racines de l'équation (1), on a :

$$(4) \qquad \sqrt{x_\infty}=a_0\sqrt{-3}, \quad \sqrt{x_s}=a_0+\epsilon^s a_1, \quad (s=0,1,2)$$

dans lesquelles ϵ est une racine cubique imaginaire de l'unité ; et

$$(5) \qquad a_0=ml\sqrt{2}, \quad a_1=-m\sqrt{2}$$

étant l le paramètre de la transformée canonique:

$$x^3+y^3+z^3+6lxyz$$

de la forme U, et m le module de la transformation. On a aussi, comme il est connu :

$$S=4m^4l\,(l^3-1), \quad T=m^6\,(8l^6+20l^3-1),$$
$$T^2-S^3=m^{12}\,(1+8l^3)^3.$$

2°. Je pose maintenant dans la relation (2) chacune des valeurs (4), l'on obtient facilement les suivantes :

$$\sqrt{X_\infty}=A_0\sqrt{-3}, \quad \sqrt{X_s}=A_0+\epsilon^s A_1,$$

étant :

$$A_0=m\,\sqrt{2}\,[al+bm^2\,(1+2l^3)],$$
$$A_1=m\sqrt{2}\,[-a+6bm^2l^2].$$

Or en supposant: $a = \frac{T}{\sqrt{S^3 - T^2}}, \quad b = -\frac{S}{\sqrt{S^3 - T^2}},$

les valeurs de A_0, A_1 deviennent:

$$A_0 = m\sqrt{2}\frac{3l}{\sqrt{-(1+8l^3)}}, \quad A_1 = m\sqrt{2}\frac{1-4l^3}{\sqrt{-(1+8l^3)}};$$

ou en posant: $\frac{A_1}{A_0} = 2L,$

on aura: (6) $6L = \frac{1-4l^3}{l},$

c'est-à-dire L sera le paramètre de la transformée canonique:

$$a^3 + \beta^3 + \gamma^3 + 6La\beta\gamma = 0$$

de la courbe de troisième classe qui porte le nom de Mr. Cayley. (Voir Salmon, "A Treatise of the Higher Plane Curves," page 191.)

En substituant les valeurs supérieures de a, b dans l'équation (2), on trouve que deux racines correspondantes des équations (1), (3) sont liées par la relation:

$$(7) \qquad X = \frac{Tx + 3S^2}{Sx - T}.$$

Ceci est réciproque et donne aussi:

$$x = \frac{TX + 3S^2}{SX - T}.$$

L'équation (3) devient dans ce cas là:

$$X^4 + 18SX^2 + 8T\,\frac{T^2 - 9S^3}{S^3 - T^2}\,X - 27S^2 = 0,$$

ou, si l'on pose: $Y = \frac{1}{27}\,\frac{(T^2 + 3S^3)^3}{S^3(S^3 - T^2)^2},$

l'on aura: $Y : Y - 1 : 1 =$

$$(X^2 + 3S)^3(X^2 + 27S) : (X^4 + 18SX^2 - 27S^2)^2 : \overline{12}^3\,S^3X^2.$$

La valeur (6) de L nous donne:

$$1 + 2L = \frac{(1-l)(1+2l)^2}{3l},$$

$$1 + 2\epsilon L = \frac{(\epsilon - l)(1 + 2\epsilon^2 l)^2}{3l},$$

$$1 + 2\epsilon^2 L = \frac{(\epsilon^2 - l)(1 + 2\epsilon l)^2}{3l};$$

on aura ainsi:

$$1 + 2L = \frac{\sqrt{X_0}}{\sqrt{X_\infty}}\sqrt{-3}, \quad 1 + 2\epsilon L = \frac{\sqrt{X_1}}{\sqrt{X_\infty}}\sqrt{-3}, \quad 1 + 2\epsilon^2 L = \frac{\sqrt{X_2}}{\sqrt{X_\infty}}\sqrt{-3},$$

et après:

$$(8)\quad 1+2l = -\left[\frac{X_0\, x_\infty}{X_\infty\, x_0}\right]^{\frac{1}{4}}\sqrt{-3},\qquad 1+2\epsilon^2 l = -\left[\frac{X_1\, x_\infty}{X_\infty\, x_1}\right]^{\frac{1}{4}}\sqrt{-3},$$

$$1+2\epsilon l = -\left[\frac{X_2\, x_\infty}{X_\infty\, x_2}\right]^{\frac{1}{4}}\sqrt{-3}.$$

3°. Un second théorème était énonçé dans cette communication à l'Académie des Sciences, sous cette forme: "L'équation dont les racines sont données par les valeurs de l'expression:

$$(9)\qquad \sqrt{Z} = \frac{1}{2S\sqrt{2}}\left[a\,(x^3-5Sx+8T) - bS\,(x^2-5S)\right]\sqrt{x},$$

correspondantes à $x = x_\infty,\ x_0,\ x_1,\ x_2$, est la suivante:

$$(10)\qquad Z^4 - 6S^{ab}Z^2 + 8T^{ab}Z - 3\,(S^{ab})^2 = 0,$$

S^{ab}, T^{ab} étant les invariants de la forme cubique $aP+bQ$, et P, Q les deux contrevariants du troisième ordre de la forme U."

Or l'on a, comme on sait,

$$\begin{aligned} S^{ab} = {} & (3S^3+T^2)\,a^4 + 16S^2Ta^3b + 6S\,(S^3+3T^2)\,a^2b^2 \\ & + 8T\,(S^3+T^2)\,ab^3 + S^2\,(5T^2-S^3)\,b^4, \\ T^{ab} = {} & T\,(9S^3-T^2)\,a^6 + 6S^2\,(3S^3+5T^2)\,a^5b + 15ST\,(5S^3+3T^2)\,a^4b^2 \\ & + 20T^3\,(7S^3+T^2)\,a^3b^3 + 15S^2T\,(S^3+7T^2)\,a^2b^4 \\ & + 6S\,(S^6+S^3T^2+6T^4)\,ab^5 + T\,(5S^6-5S^3T^2+8T^4)\,b^6; \end{aligned}$$

par conséquent, si l'on pose:

$$a = \frac{T}{(S^3-T^2)^{\frac{1}{2}}},\quad b = \frac{-S}{(S^3-T^2)^{\frac{1}{2}}},$$

on trouve:

$$S_{ab} = -1,\quad T_{ab} = -\frac{T}{\sqrt{S^3-T^2}},$$

et l'équation en Z devient là:

$$Y : Y-1 : 1 =$$
$$= (Z^2-1)^3(Z^2-9) : (Z^4+6Z^2-3)^2 : 4^3.Z^2,$$

étant dans ce cas:

$$Y = \frac{S^3}{S^3-T^2}.$$

Enfin, en observant que de la valeur (9) de $\sqrt{Z}$ on déduit la relation:

$$(11)\qquad \sqrt{X} = Z\sqrt{x},$$

on pourra substituer aux (8) les suivantes:

$$1+2l = -\sqrt{\frac{Z_0}{Z_\infty}}\cdot\sqrt{-3},\qquad 1+2\epsilon^2 l = -\sqrt{\frac{Z_1}{Z_\infty}}\cdot\sqrt{-3},$$

$$1+2\epsilon l = -\sqrt{\frac{Z_2}{Z_\infty}}\cdot\sqrt{-3};$$

ou, en posant: (12) $\sqrt{Z_\infty} = -a_0\sqrt{-3}, \quad \sqrt{Z_\epsilon} = a_0 + \epsilon^{2\epsilon} a_1,$

on aura :

$$l = \frac{1}{2}\frac{a_1}{a_0},$$

à laquelle on peut ajouter l'autre qu'on déduit des équations (5),

$$l = -\frac{a_0}{a_1}.$$

4°. Soient k le module des fonctions elliptiques, λ, μ le module transformé et le multiplicateur dans la transformation du troisième ordre. Si dans l'équation (1) on pose :

$$x = \mu\sqrt{S},$$

et l'on suppose déterminé le module k par la relation :

$$\frac{T}{S\sqrt{S}} = 1-2k^2,$$

on obtient l'équation du multiplicateur :

$$\mu^4 - 6\mu^2 + 8(1-2k^2)\mu - 3 = 0.$$

Soit : $$X = \nu\sqrt{S},$$

la relation (7) nous donne :

$$\nu = \frac{(1-2k^2)\mu + 3}{\mu - (1-2k^2)},$$

de laquelle on déduit facilement que :

$$\nu = \frac{\mu(\mu^2-9)}{1-\mu^2};$$

mais l'équation du multiplicateur peut s'écrire des deux manières suivantes :

$$(\mu-1)^3(\mu+3) = 16k^2\mu, \quad (\mu+1)^3(\mu-3) = -16k'^2\mu;$$

en conséquence on aura : $\nu = 4^4 k^2 k'^2 \dfrac{\mu^3}{(1-\mu^2)^4}.$

Si maintenant on se rappelle les formules connues :

$$\sqrt[4]{\frac{\lambda}{k}} = -2\frac{\sqrt{k}}{1-\mu}, \quad \sqrt[4]{\frac{\lambda'}{k'}} = -2\frac{\sqrt{k'}}{1+\mu},$$

on voit que: $$\nu = \mu^3\frac{\lambda\lambda'}{kk'}.$$

L'équation du quatrième degré en ν sera :

$$Y : Y-1 : 1 =$$

$$(\nu^2+3)^3(\nu^2+27) : (\nu^4+18\nu^2-27)^2 : \overline{12}^3 . \nu^2,$$

étant dans ce cas : $$Y = \frac{4}{27}\frac{(1-k^2k'^2)^3}{k^4k'^4}.$$

En indiquant avec K, Λ les fonctions complètes de première espèce, la valeur supérieure de ν donne les suivantes :

$$\sqrt{\nu_\infty} = -3\sqrt{\frac{\Lambda_\infty^3\,\lambda_\infty\,\lambda'_\infty}{K^3 kk'}}\cdot\sqrt{-3}, \qquad \sqrt{\nu_s} = \sqrt{\frac{\Lambda_s^3\,\lambda_s\,\lambda'_s}{K^3 kk'}};$$

or on sait que, en posant $q = e^{-\pi\frac{k'}{k}}$, on a : *

$$\sqrt{\frac{2k^3kk'}{\pi^3}} = q^{\frac14}\,\Pi\,(1-q^{2m})^3 = F(q),$$

et par conséquent :

$$\sqrt{\nu_\infty} = -3\,\frac{F(q^3)}{F(q)}\,\sqrt{-3}, \qquad \sqrt{\nu_s} = \frac{F(\epsilon^s q^{\frac13})}{F(q)};$$

mais par une autre formule des "Fundamenta Nova," p. 185, on a :

$$F(q) = q^{\frac14}\,[1-3q^2+5q^6-7q^{12}+9q^{20}-11q^{30}+\ldots]$$

et $$F(\epsilon^s q^{\frac13}) = \epsilon^s q^{\frac{1}{12}}\,[1+5q^2-7q^4-11q^{10}+\ldots]-3F(q^3);$$

on pourra donc écrire :

$$\sqrt{\nu_\infty} = A_0\,\sqrt{-3}, \qquad \sqrt{\nu_s} = A_0 + A_1 s,$$

étant : $A_0 = -3\,\dfrac{F(q^3)}{F(q)}, \quad A_1 = \dfrac{q^{\frac{1}{12}}}{F(q)}\,[1+5q^2-7q^4-11q^{10}+\ldots],$

et on obtiendra pour L la valeur : †

$$L = \tfrac12\,\frac{A_1}{A_0} = -\,\frac{1}{6q^{\frac23}}\,\frac{1+5q^2-7q^4-11q^{10}+13q^{14}\ldots}{1-3q^6+5q^{18}-7q^{36}+\ldots}.$$

5°. De la même manière l'équation (11) donne :

$$Z = \sqrt{\frac{\nu}{\mu}} = \mu\,\sqrt{\frac{\lambda\lambda'}{kk'}},$$

mais l'on a : $$\left(\frac{K^2 kk'}{\Pi^2}\right)^{\frac14} = q^{\frac18}\,\frac{\Pi\,(1-q^{2m})}{\Pi\,(1+q^{2m-1})} = f(q)$$

et : $$f(q) = q^{\frac18}\,[1-q-q^3+q^6+q^{10}-q^{15}-q^{21}+\ldots];$$

on déduira donc des équations (12) :

$$a_0 = -\,\frac{f(q^3)}{f(q)}, \quad a_1 = \frac{q^{\frac{1}{24}}}{f(q)}\,[1-q+q^2-q^5-q^7+q^{12}-q^{15}+\ldots],$$

et l'on aura : $$l = \tfrac12\,\frac{a_1}{a_0} = -\,\frac{1}{2q^{\frac13}}\,\frac{1-q+q^2-q^5-q^7+\ldots}{1-q^3-q^9+q^{18}+\ldots},$$

---

* "Fundamenta Nova," page 89. Cayley, "An Elementary Treatise of Elliptic Functions," page 287.

† Mr. le Prof. Klein avait déjà obtenu cette formule par d'autres considérations dans son beau Mémoire, "Ueber die Transformation der elliptischen Functionen etc." (*Math. Ann.*, Bd. xiv.) Voir aussi un récent travail de Mr. Bianchi (élève de Mr. Klein) dans le même Journal, Bd. xvii.

ou aussi : $l = -\frac{a_0}{a_1} = -\frac{1}{2q^{\frac{1}{8}}} \frac{1+2q^3+2q^{12}+2q^{27}+\dots}{1+q+q^5+q^8+q^{16}+q^{21}+\dots}$.

6. Le caractère *tétraèdrique*, selon la dénomination des MM. Schwarz et Klein, des équations du quatrième degré (3), (10), peut en général se déduire des considérations suivantes. En posant :

$$f(a, b) = S_{ab},$$

soient : $h = \frac{1}{2}(ff)_2$, $\theta = 2(fh)$, $g_2 = \frac{1}{2}(ff)_4$, $g_3 = \frac{1}{3}(fh)_4$

les deux covariants et les deux invariants de la forme binaire f. On trouve : $h = (S^3 - T^2)(a^4 - 6Sa^2b^2 - 8Tab^3 - 3S^2b^4)$,

$$\theta = -(S^3 - T^2)\, T_{ab}, \quad g_2 = 0, \quad g_3 = -4(S^3 - T^2)^2,$$

et par ces valeurs :

$$Y = -\frac{4h^3}{g_3 f^3} = \frac{(a^4 - 6Sa^2b^2 - 8Tab^3 - 3S^2b^4)^3}{S_{ab}^3}(S^3 - T^2);$$

mais on a : $S_{ab}^3 - T_{ab}^2 = (S^3 - T^2)(a^4 - 6Sa^2b^2 - 8Tab^3 - 3S^2b^4)^3$,

donc : $$Y = \frac{S_{ab}^3 - T_{ab}^2}{S_{ab}^3},$$

et l'équation (2) pourra s'écrire :

$$Y : Y-1 : 1 =$$

$$(X^2 - S_{ab})^3 (X^2 - 3S_{ab}) : (X^4 - 6S_{ab}X^2 - 3S_{ab}^2)^2 : -8^2 S_{ab}^3 X^2,$$

et analoguement pour l'autre.

Supplement on Binomial Biordinals.

By Sir JAMES COCKLE, M.A., F.R.S., &c.

[*Read February* 10*th*, 1881.]

33. The general form of the second class of cases mentioned in Art. 5 (Vol. xi., p. 123), is characterized by

$$m = (\tfrac{3}{2}i + \tfrac{1}{8})(\tfrac{3}{2}i + \tfrac{5}{8}),$$

where i is an integer or zero. Here

$$1 + 16m = 1 + (6i + \tfrac{1}{2})(6i + \tfrac{5}{2}) = (6i + \tfrac{3}{2})^2,$$

and the symbolical terordinal becomes

$$[D]^3 y - (D - \tfrac{5}{2})(D + 3i - \tfrac{5}{2} + \tfrac{3}{4})(D - 3i - \tfrac{5}{2} - \tfrac{3}{4})\, x^3 y = 0,$$

which is, or by the processes of Boole (see Art. 11) can be transformed into,

$$[D]^3 v-(D-\tfrac{1}{4})\,(D-\tfrac{10}{4})\,(D-\tfrac{13}{4})\,x^3 v=0,$$

the symbolical form of the differential resolvent of the quartic $v^4-4v+3x=0$ (see Mr. Harley's papers, cited in Art. 14).

34. This general form is therefore finitely soluble (see Arts. 15 to 24).

35. Any binomial biordinal whatever is of, or may be transformed into, the form

$$(1-x^3)\,z''+\left(\frac{\lambda}{x}+\mu x^2\right)z'+\left(\frac{\nu}{x^2}+\rho x\right)z=0;$$

and if we put

$$1-x^3=X,\quad \lambda+\mu x^3=P,\quad \nu+\rho x^3=Q,$$

and write the biordinal thus,

$$z''+\frac{P}{xX}z'+\frac{Q}{x^2X}z=0,$$

the auxiliary terordinal of Art. 7 will be

$$z'''+3\frac{P}{xX}z''+qz'+rz=0,$$

wherein

$$q=\left(\frac{P}{xX}\right)'+2\left(\frac{P}{xX}\right)^2+4\frac{Q}{x^2X}=\frac{4Q+xP'}{x^2X}+\frac{2P^2-(xX)'P}{x^2X^2},$$

$$r=2\left\{\frac{xQ'}{x^3X}+Q\,\frac{2xP-(x^2X)'}{x^4X^2}\right\};$$

but $(xX)'=X-3x^3=4X-3$ and $(x^2X)'=2x-5x^4=x\,(5X-3)$, consequently

$$q=\frac{4\,(Q-P)+xP'}{x^2X}+\frac{P\,(2P+3)}{x^2X^2},\quad r=2\left\{\frac{xQ'-5Q}{x^3X}+\frac{(2P+3)\,Q}{x^3X^2}\right\}.$$

36. For the biordinal

$$Xx^2z''+Pxz'+Qz=0,$$

we have, therefore, the terordinal

$$Xx^3y'''+3Px^2y''+\left\{4\,(Q-P)+xP'+\frac{P\,(2P+3)}{X}\right\}xy'$$
$$+2\left\{xQ'-5Q+\frac{(2P+3)\,Q}{X}\right\}y=0.$$

37. The terordinal will be binomial if P and Q are each divisible by X. But this requires that $\lambda+\mu$ and $\nu+\rho$ should both vanish. Our results in such case would be useless, for the biordinal would become $(1-x^3)\,(x^2z''+\lambda xz'+\nu z)=0$, which is soluble as it stands.

38. The terordinal is also binomial if $2P+3$ be divisible by X, that is, if the single condition $2\mu+3=-2\lambda$ or $\lambda+\mu+\frac{3}{2}=0$ be satisfied. We then obtain the relation

$$2P+3=2(\lambda+\mu)+3-2\mu X=-2\mu X,$$

and the binomial terordinal becomes

$$Xx^3y'''+3Px^2y''+\{4(Q-P)+xP'-2\mu P\}xy' \\ +2\{xQ'-(5+2\mu)Q\}y=0,$$

or, substituting for X, P, P', Q, and Q' and reducing,

$$(1-x^3)x^3y'''+3(\lambda+\mu x^3)x^2y''+\{2\lambda^2-\lambda+4\nu \\ -(2\mu^2+\mu-4\rho)x^3\}xy'+2\{2(\lambda-1)\nu-2(\mu+1)\rho x^3\}y=0.$$

39. In this equation μ stands for $-(\lambda+\frac{3}{2})$. The result may be represented by

$$f(y,x,\lambda,\nu)-x^3f(y,x,-\mu,-\rho)=0.$$

40. The symbolical form of $f(y,x,\lambda,\nu)$ is

$$(D+\lambda-1)\{(D+\lambda-1)^2-(\lambda-1)^2+4\nu\}y,$$

consequently that of $f(y,x,-\mu,-\rho)$ is

$$(D-\mu-1)\{(D-\mu-1)^2-(\mu+1)^2-4\rho\}y,$$

and that of the terordinal is

$$y-\frac{(D-\mu-4)\{(D-\mu-4)^2-(\mu+1)^2-4\rho\}}{(D+\lambda-1)\{(D+\lambda-1)^2-(\lambda-1)^2+4\nu\}}x^3y=0.$$

41. If, as in Art. 26, we change the D in the numerator to $D+3$, then, replacing μ by its value $-\lambda-\frac{3}{2}$ (eliminating μ), we get

$$v-\frac{(D+\lambda+\frac{1}{2})\{D+\lambda+\frac{1}{2}+\sqrt{(\lambda+\frac{1}{2})^2+4\rho}\}\{D+\lambda+\frac{1}{2}-\sqrt{(\lambda+\frac{1}{2})^2+4\rho}\}}{(D+\lambda-1)\{D+\lambda-1+\sqrt{(\lambda-1)^2-4\nu}\}\{D+\lambda-1-\sqrt{(\lambda-1)^2-4\nu}\}} \\ \times x^3v=0.$$

42. In verification take the biordinal of Art. 2. Here $\lambda=0$, $\mu=-\frac{3}{2}$, $\nu=0$, $\rho=m$. Take that of Art. 24. Here $\lambda=0$, $\mu=-\frac{3}{2}$, $\nu=0$, $\rho=\frac{5}{64}$. Take that of Art. 25. Here $\lambda=-(n+\frac{1}{2})$, $\mu=n-1$, $\nu=-\rho=-m$. All these examples are in accord with the result of Art. 41.

43. For, divided by $[D]^3$, the terordinals of Arts. 10 and 33 respectively are

$$y-\frac{(D-\frac{5}{2})(D-\frac{5}{2}+\sqrt{\frac{1}{4}+4m})(D-\frac{5}{2}-\sqrt{\frac{1}{4}+4m})}{[D]^3}x^3y=0$$

and
$$v-\frac{(D-\frac{7}{4})(D-\frac{10}{4})(D-\frac{13}{4})}{[D]^3}x^3v=0;$$

and a change of D into $D+3$ in the numerators, accompanied by the

substitutions indicated severally in Art. 42 as appropriate to the first two examples, will identify these terordinals with that of Art. 41. In the third case, substitution at once changes the last terordinal of Art. 26 to that of Art. 41.

44. In further verification, add a term $\frac{\nu+m}{x^2X}z$ to the sinister of the biordinal of Art. 25. The effect is (see Arts. 7 and 25) to augment q by $4\frac{\nu+m}{x^2X}$, and r by

$$2\left(\frac{\nu+m}{x^2X}\right)'+4(\nu+m)\left(\frac{1-n}{x}-\frac{\frac{3}{4}}{xX}\right)\frac{1}{x^2X},$$

viz., by

$$2(\nu+m)\left\{\frac{-(x^2X)'+2(1-n)xX-3x}{(x^2X)^2}\right\}, \text{ or } -\frac{2(2n+3)(\nu+m)}{x^3X},$$

and these augmentations change the coefficient of xy' in the equation at the head of p. 130 into $2n^2+3n+1+4\nu-(2n^2-3n+1-4m)x^3$, and that of y into $-2(2n+3)\nu-4mnx^3$. The last set of substitutions in Art. 42 will identify the terordinal of Art. 38 with the augmented terordinal. I remark that

$$-4mn = m(4\lambda+2) = -4(\mu+1)\rho.$$

45. The symbolical form of the biordinal of Art. 35 is

$$z-\frac{(D-3)^2-(\mu+1)(D-3)-\rho}{D^2+(\lambda-1)D+\nu}x^3z=0,$$

or, changing the D in the numerator into $D+3$, and decomposing into factors

$$z-\frac{\left(D-\frac{\mu+1}{2}+\frac{1}{2}\sqrt{(\mu+1)^2+4\rho}\right)\left(D-\frac{\mu+1}{2}-\frac{1}{2}\sqrt{(\mu+1)^2+4\rho}\right)}{\left(D+\frac{\lambda-1}{2}+\frac{1}{2}\sqrt{(\lambda-1)^2-4\nu}\right)\left(D+\frac{\lambda-1}{2}-\frac{1}{2}\sqrt{(\lambda-1)^2-4\nu}\right)}x^3z=0,$$

or, replacing μ by $-\lambda-\frac{3}{2}$, *i.e.*, $\mu+1$ by $-(\lambda+\frac{1}{2})$,

$$z-\frac{\{D+\frac{1}{2}(\lambda+\frac{1}{2})+\frac{1}{2}\sqrt{(\lambda+\frac{1}{2})^2+4\rho}\}\{D+\frac{1}{2}(\lambda+\frac{1}{2})-\frac{1}{2}\sqrt{(\lambda+\frac{1}{2})^2+4\rho}\}}{\{D+\frac{1}{2}(\lambda-1)+\frac{1}{2}\sqrt{(\lambda-1)^2-4\nu}\}\{D+\frac{1}{2}(\lambda-1)-\frac{1}{2}\sqrt{(\lambda-1)^2-4\nu}\}}\times x^3z=0.$$

46. Call this biordinal

$$z-\frac{(D-A_1)(D-A_2)}{(D-B_1)(D-B_2)}x^3z=0,$$

then the auxiliary terordinal of Art. 41 may be written

$$v-\frac{(D-2A_1)(D-A_1-A_2)(D-2A_2)}{(D-2B_1)(D-B_1-B_2)(D-2B_2)}x^3v=0.$$

47. Call a the root of the factor $D-a$. Then the terordinal is derived from the biordinal by doubling the roots and inserting both in the numerator and denominator a middle factor, whereof the root is an arithmetical mean between the doubled roots.

48. To facilitate comparison with Boole's canons, change, in the biordinal only, x^3 into x^2. We thus get

$$z-\frac{(D-\alpha_1)\,(D-\alpha_2)}{(D-\beta_1)\,(D-\beta_2)}x^2z=0,$$

the α's and β's being connected with the A's and B's by the four relations $\quad 3\alpha_1,\,3\alpha_2=2A_1,\,2A_2,$ and $3\beta_1,\,3\beta_2=2B_1,\,2B_2$;

and the terordinal may now be written

$$v-\frac{(D-3\alpha_1)\,\{D-\frac{3}{2}\,(\alpha_1+\alpha_2)\}\,(D-3\alpha_2)}{(D-3\beta_1)\,\{D-\frac{3}{2}\,(\beta_1+\beta_2)\}\,(D-3\beta_2)}x^3v=0;$$

but, inasmuch as the x is supposed to remain unchanged in the terordinal, we must, in applying formulæ derived therefrom to the biordinal of this Article, make a preliminary change of x^3 into x^2. There is, however, this advantage in the last transformation, viz., that we can exhibit the conditions of solubility discussed in these papers under forms resembling the conditions of Boole.

49. Since $A_1+A_2=\mu+1$, and $B_1+B_2=1-\lambda$, therefore the condition (Art. 38) $\lambda+\mu+\frac{3}{2}=0$ may be written $A_1+A_2-B_1-B_2+\frac{3}{2}=0$, which, multiplied into 2, becomes, on substitution,

$$3(\alpha_1+\alpha_2-\beta_1-\beta_2+1)=0, \quad \text{or} \quad \alpha_1+\alpha_2-\beta_1-\beta_2=-1,$$

or, say, $\alpha_1+\alpha_2-\beta_1-\beta_2=$ an odd integer; for, by Boole's algorithm, all, any, or either of the D's in the biordinal of Art. 48 can be changed into $D+2i$. This is one of Boole's conditions, and whenever it is satisfied the auxiliary terordinal is binomial.

50. But, in order that the terordinal should break up into primordinals, in which event the biordinal is of, or reducible to, the form of Schwarz (see Arts. 12 and 24), one of the two congruences $-A_1+A_2\equiv\pm1$ (mod. 3) and one of the two $-B_1+B_2\equiv\pm1$ (mod. 3) must be satisfied. This is equivalent to saying that we must have $-\alpha_1+\alpha_2=2i+\frac{2}{3}$ or $2i-\frac{2}{3}$, and also $-\beta_1+\beta_2=2i_2+\frac{2}{3}$ or $2i_2-\frac{2}{3}$. In other words, the biordinal is soluble when the following three conditions are satisfied, viz., $\alpha_1+\alpha_2-\beta_1-\beta_2$ is an odd integer, and $\alpha_1-\alpha_2$ and $\beta_1-\beta_2$ are each of one (not necessarily the same) of the two forms $2\,(i\pm\frac{1}{3})$, i being any integer, or 0.

The case in which two symbolical factors disappear from the numerator and two from the denominator of the terordinal may be briefly dealt with. Taking $\alpha_1+\alpha_2-\beta_1-\beta_2=$ an odd integer as the fundamental

condition, viz., that the terordinal may be binomial, the middle factors cannot cancel one another, for $\frac{3}{2}(\alpha_1+\alpha_2)-\frac{3}{2}(\beta_1+\beta_2)\equiv 0 \pmod{3}$ contradicts the fundamental condition. If the first factors so cancel, we have $3\alpha_1-3\beta_1\equiv 0 \pmod{3}$, or $\alpha_1-\beta_1=$ an integer. If that integer be even, a factor disappears from the biordinal; so, too, if it be odd, for then, in virtue of the fundamental condition, $\alpha_2-\beta_2$ is even. We have a similar result if the last factors mutually cancel, and, generally, the condition $3(\alpha-\beta)\equiv 0 \pmod{3}$ combined with the fundamental condition indicates a disappearing factor in the biordinal. If a middle factor cancels one of the others, we have a result of the form $\frac{3}{2}(\alpha_1+\alpha_2)-3\beta\equiv 0 \pmod{3}$, whence $\alpha_1+\alpha_2-2\beta$ is even, which combined with the fundamental condition gives $\beta_1-\beta_2$ odd, and the biordinal is soluble as it stands. So, too, if $\frac{3}{2}(\beta_1+\beta_2)-3\alpha\equiv 0 \pmod{3}$.

51. For the class of cases dealt with in Art. 33, we have $-A_1+A_2\equiv\pm\frac{3}{4} \pmod{3}$, or say $-\alpha_1+\alpha_2=2i_3\pm\frac{1}{2}$, and, as in Art. 50, $-\beta_1+\beta_2=2(i\pm\frac{1}{3})$. And since, by the change of x into x^{-1}, and by Boole's algorithm, we may change α into $-\beta$, and β into $-\alpha$, this class of cases may be said to be soluble when $\alpha_1+\alpha_2-\beta_1-\beta_2$ is an odd integer, and one of the two quantities $\alpha_1-\alpha_2$ and $\beta_1-\beta_2$ is of the form $2(i\pm\frac{1}{3})$ and the other of the form $2i_3\pm\frac{1}{2}$ or $j\pm\frac{1}{2}$. Both, or either, $\pm$ may be replaced by $\mp$.

52. The numerator and denominator of a binomial auxiliary terordinal are arithmetical progressions whereof the respective differences are $\frac{3}{2}(\alpha_1-\alpha_2)$ and $\frac{3}{2}(\beta_1-\beta_2)$.

53. The above conditions are sufficient, but are necessary only in a qualified sense. There exist transformations, unnoticed by Boole, whereby we may interchange the three quantities $\beta_1-\beta_2$, $\alpha_1+\alpha_2-\beta_1-\beta_2$, $\alpha_1-\alpha_2$ in any way that we please. Represent these quantities by I, U, J respectively. Then, remembering that the biordinal form of Art. 45 has its full complement of symbolical factors (viz., four, two in the numerator and two in the denominator), the transformation of I, U, J into $-J, U, I$ can be effected. For the change of x into x^{-1}, *i.e.*, of D into $-D$, and the other operations of Boole (pp. 428–9; *Supplementary Volume*, 1865, pp. 184–6) will enable us to change α into $-\beta$ and β into $-\alpha$.

54. Since, in Arts. 50 and 51, the integers i, j may be positive or negative, and the $\pm$ may be $\mp$, the conditions of solubility, if satisfied for the given, are satisfied for the transformed equation.

55. I proceed to the transformation of I, U, J into $-U, I, J$. Let $\dfrac{(\dot{D}-\dot{\alpha}_1)(\dot{D}-\dot{\alpha}_2)}{(\dot{D}-\dot{\beta}_1)(\dot{D}-\dot{\beta}_2)}=\phi(\dot{D})$, and $w+h^2\phi(\dot{D})\,x^k w=0$. Then this most

general form can be transformed (see Boole, pp. 418 *et seq.*; Supplement, pp. 184 *et seq.*) into $u-\phi(D)\,x^2u=0$, all dots being omitted in $\phi(D)$. And this last biordinal can be transformed into

$$D(D+\beta_1-\beta_2)\,v-(D+\beta_1-\alpha_1)(D+\beta_1-\alpha_2)\,x^2v=0,$$

or, putting $\beta_2-\beta_1=B,\ \alpha_1-\beta_1=A_1,\ \alpha_2-\beta_1=A_2,$

$$D(D-B)\,v-(D-A_1)(D-A_2)\,x^2v=0.$$

56. Let $x=\sqrt{1+t^2}$, then

$$D=x\frac{d}{dx}=\left(t+\frac{1}{t}\right)\frac{d}{dt},$$

or, putting $t+\dfrac{1}{t}=T,\qquad D=T\dfrac{d}{dt}.$

Eliminating D and x, we get

$$T\frac{d}{dt}\left(T\frac{d}{dt}-B\right)v-\left(T\frac{d}{dt}-A_1\right)\left(T\frac{d}{dt}-A_2\right)(1+t^2)\,v=0,$$

or

$$\left\{T^2\frac{d^2}{dt^2}+(T'-B)\,T\frac{d}{dt}\right\}v$$

$$-\left\{T^2\frac{d^2}{dt^2}+(T'-A_1-A_2)\,T\frac{d}{dt}+A_1A_2\right\}(1+t^2)\,v=0,$$

where $T'=\dfrac{dT}{dt}$.

57. But this result reduces to

$$T(A_1+A_2-B)\frac{dv}{dt}-A_1A_2(1+t^2)\,v$$

$$-\left\{T^2\frac{d^2}{dt^2}+(T'-A_1-A_2)\,T\frac{d}{dt}\right\}t^2v=0,$$

which becomes binomial on dividing out T, for then we have, putting $A_1+A_2=S$,

$$(S-B)\frac{dv}{dt}-A_1A_2tv-\left\{T\frac{d^2}{dt^2}+\left(1-\frac{1}{t^2}-S\right)\frac{d}{dt}\right\}t^2v=0.$$

58. Let $\Delta=t\dfrac{d}{dt}$; then we have

$$\frac{S-B}{t}\Delta v-A_1A_2tv-\left\{\left(\frac{1}{t}+\frac{1}{t^3}\right)\Delta(\Delta-1)+\left(\frac{1-S}{t}-\frac{1}{t^3}\right)\Delta\right\}t^2v=0;$$

whence, multiplying into t, and reducing,

$$\{(S-B)\,\Delta-\Delta(\Delta+2)\}\,v-\{\Delta(\Delta-1)+(1-S)\,\Delta+A_1A_2\}\,t^2v=0,$$

or

$$\Delta(\Delta+B-S+2)\,v+\{\Delta^2-S\Delta+A_1A_2\}\,t^2v=0.$$

59. Take a new x, $=t\sqrt{-1}$, and restore A_1+A_2; we obtain

$$D(D+B-A_1-A_2+2)\,v-(D-A_1)(D-A_2)\,x^2v=0,$$

and hence, changing D into $D-\beta_1$,

$$(D-\beta_1)(D-a_1-a_2+\beta_2+2)\,V-(D-a_1)(D-a_2)\,x^2V=0.$$

60. We have thus changed β_2 into $a_1+a_2-\beta_2-2$, and we can therefore (2 being even) change β_2 into $a_1+a_2-\beta_2$ without affecting the a's or β_1. But this change implies that of $\beta_1-\beta_2$ into $\beta_1+\beta_2-a_1-a_2$, that is, into $-U$. To avoid the use of negative signs, the entrance of which does not (see Art. 54) materially affect our conditions, I shall suppose our transformations applied to I^2, U^2, and J^2. And we have seen that we may change I^2, U^2, J^2 into J^2, U^2, I^2, and also into U^2, I^2, J^2.

Calling the first transformation f and the second ϕ, we may write $f(I^2, U^2, J^2)=(J^2, U^2, I^2)$ and $\phi(I^2, U^2, J^2)=(U^2, I^2, J^2)$. It follows that $\phi f(I^2, U^2, J^2)=(U^2, J^2, I^2)$, $f\phi(I^2, U^2, J^2)=(J^2, I^2, U^2)$, and $\phi f\phi(I^2, U^2, J^2)=(I^2, J^2, U^2)$. These are all the transformations of the kind.

61. Recurring to Arts. 7 and 8, and using an unsuffixed b connected with b_1 and b_2 by the relation

$$b=b_2-\tfrac{1}{2}b_1'-\tfrac{1}{4}b_1^2,$$

and a quantity B defined by $B=e^{\int b_1 dx}$, we may write the terordinal in the form

$$y'''+3\frac{B'}{B}y''+\left(4b+3\frac{B''}{B}\right)y'+\left(2b'+4\frac{B'}{B}b+\frac{B'''}{B}\right)y=0,$$

which, multiplied into B, takes the form

$$(By)'''+4b\,(By)'+2b'\,(By)=0,$$

or, putting $By=Y$,

$$Y'''+4bY'+2b'Y=0.$$

62. One first integral of this terordinal (compare Art. 8) is

$$YY''-\tfrac{1}{2}(Y')^2+2bY^2=c_1,$$

and it is a property of this terordinal that any of its particular solutions is also one of its integrating factors. Thus, if $y=X$, X being a function of x, be a particular solution, we have a second first integral, also involving an arbitrary constant, viz.,

$$XY''-X'Y'+(X''+4bX)\,Y=c_2.$$

63. This equation, differentiated, yields

$$XY'''+4bXY'+(X'''+4bX'+4b'X)\,Y=0,$$

which reduces to

$$X(Y''' + 4bY' + 2b'Y) + Y(X''' + 4bX' + 2b'X) = 0,$$

or to $Y''' + 4bY' + 2b'Y = 0$, for, $y = X$ being a particular solution, the last bracketed quantity vanishes.

64. Combining the two first integrals, we get

$$\tfrac{1}{2}X(Y')^2 - X'YY' + (X'' + 2bX)Y^2 = c_2 Y - c_1 X,$$

or $$\left(Y' - X'\frac{Y}{X}\right)^2 - (X'^2 - 2XX'' - 4bX^2)\frac{Y^2}{X^2} - 2c_2\frac{Y}{X} + 2c_1 = 0;$$

and if X involve no arbitrary constant, then $X'^2 - 2XX'' - 4bX^2$, which, differentiated, vanishes [in virtue of its becoming $-2X(X''' + 4bX' + 2b'X)$], is a constant, say k^2. Give to the arbitrary constants c_1, c_2 the respective values $-\frac{1}{2}k^2$, $+k^2$. Then

$$\left(Y' - X'\frac{Y}{X}\right)^2 = k^2\left(\frac{Y}{X} + 1\right)^2, \quad Y' = X'\frac{Y}{X} \pm k\left(\frac{Y}{X} + 1\right),$$

and $$Y = X\left(Ce^{\pm k\int\frac{dx}{X}} - 1\right).$$

The process is illusory when $k = 0$, but is always effective when the particular solution of the terordinal does *not* satisfy the given biordinal.

65. The complete solution of the terordinal is therefore

$$X\left(C_1 + C_2 e^{k\int\frac{dx}{X}} + C_3 e^{-k\int\frac{dx}{X}}\right) = Y,$$

a solution which will be sufficiently verified when we come to verify the coresponding solution of the biordinal.

66. If we multiply the generalized biordinal of Art. 7 into $e^{\frac{1}{2}\int b_1 dx}$, the product may be put under the form

$$\left(e^{\frac{1}{2}\int b_1 dx} z\right)'' + b\left(e^{\frac{1}{2}\int b_1 dx} z\right) = \frac{c}{\left(e^{\frac{1}{2}\int b_1 dx} z\right)^3},$$

the unsuffixed b being defined in Art. 61; or, putting $e^{\frac{1}{2}\int b_1 dx} z = Z$,

$$Z'' + bZ = \frac{c}{Z^3}.$$

67. If the three constants C_1, C_2, C_3 of Art. 65 be connected by $4C_2C_3 = C_1^2$, then will

$$Z = \sqrt{2Y} = \sqrt{X}\left(Fe^{\frac{1}{2}k\int\frac{x}{X}} + Ge^{-\frac{1}{2}k\int\frac{dx}{X}}\right),$$

wherein F and G are arbitrary constants, be the complete solution of the biordinal in Z, which arises when c is put $= 0$.

68. In verification of this take the particular solution

$$(Z) = \sqrt{X} e^{\frac{1}{2}k\int\frac{dx}{X}}.$$

Then
$$(Z)' = \tfrac{1}{2}\left(\frac{X'}{X}+\frac{k}{X}\right)(Z);$$

$$(Z)'' = \tfrac{1}{2}\left(\frac{X''}{X}-\frac{X'^2}{X^2}-\frac{kX'}{X^2}\right)(Z)+\tfrac{1}{4}\left(\frac{X'}{X}+\frac{k}{X}\right)^2(Z)$$
$$= \left(\tfrac{1}{2}\frac{X''}{X}-\tfrac{1}{4}\frac{X'^2}{X^2}+\tfrac{1}{4}\frac{k^2}{X^2}\right)(Z),$$

and
$$(Z)''+b(Z) = \left(\tfrac{1}{2}\frac{X''}{X}-\tfrac{1}{4}\frac{X'^2}{X^2}+\tfrac{1}{4}\frac{k^2}{X^2}+b\right)(Z);$$

but the dexter is $\tfrac{1}{4}\dfrac{1}{X^2}(2XX''-X'^2+4bX^2+k^2)(Z)$, and the coefficient of (Z) vanishes (see Art. 64). And since $k^2=(-k)^2$ the whole solution is verified.

69. I remark that, if we can so assign the arbitrary constant c as to obtain a particular solution of the generalized biordinal in Z, we can completely integrate $Z''+bZ=0$. Let $Z=\zeta$ be such a particular solution. Then $Z = \left(Fe^{\int\frac{\sqrt{(-c)}}{\zeta^2}dx}+Ge^{-\int\frac{\sqrt{(-c)}}{\zeta^2}dx}\right)\zeta$

is the complete solution of $Z''+bZ=0$. For, taking the particular solution $(Z)=\zeta e^{\int\frac{\sqrt{(-c)}}{\zeta^2}dx}$, we get

$$(Z)' = \left(\frac{\zeta'}{\zeta}+\frac{\sqrt{-c}}{\zeta^2}\right)(Z);$$

$$(Z)'' = \left(\frac{\zeta''}{\zeta}-\frac{\zeta'^2}{\zeta^2}-\frac{2\sqrt{-c}\,\zeta'}{\zeta^3}\right)(Z)+\left(\frac{\zeta'}{\zeta}+\frac{\sqrt{-c}}{\zeta^2}\right)^2(Z) = \left(\frac{\zeta''}{\zeta}-\frac{c}{\zeta^4}\right)(Z),$$

and
$$(Z)''+b(Z) = \left(\frac{\zeta''}{\zeta}+b-\frac{c}{\zeta^4}\right)(Z),$$

and the dexter vanishes, for $Z=\zeta$ solves the generalized biordinal in Z. And the radical may be taken with either sign, so that the whole solution is verified.

Some Applications of Conjugate Functions.
By E. J. Routh, F.R.S.

[*Read Feb.* 10*th*, 1881.]

Contents.

Part I. *Motion of Membranes.*

Arts. 1—5. Motion of a heterogeneous membrane deduced from that of a homogeneous membrane.

Arts. 6—8. Geometrical statements concerning the distribution of density.

Art. 9. Curves of equal density.

Arts. 11—16. Examples showing how to write down the periods of vibration of membranes bounded by two eccentric circles, or by two concentric circles, and two radii vectores, &c. &c., but with suitable laws of density.

Arts. 17, 18. A source of vibration disturbing a membrane.

Part II. *Motion of a Fluid.*

Arts. 19, 20. Some preliminary remarks.

Arts. 21—23. Three propositions:

(1) The kinetic energies over corresponding areas are equal.

(2) Sources and vortices transform into sources and vortices of equal strengths.

(3) The current function of a transformed vortex is obtained from that of the original vortex by subtracting a known function of the coordinates of that vortex. This is not the current function of the motion of the fluid.

Arts. 24—26. A fourth proposition, to find the proper formulæ of transformation.

Art. 27. *Example* 1.—Motion in a corner.

Art. 28. *Example* 2.—Motion when the boundaries are parallel to the axes.

Arts. 29—31. *Example* 3.—Motion round an obstacle of any form with an application to the case of an elliptic obstacle.

Part I.

1. If we have two variables ξ, η connected with x, y, so that

$$\xi+\eta\sqrt{-1}=f(x+y\sqrt{-1}),$$

then $\dfrac{d\xi}{dx}=\dfrac{d\eta}{dy}$ and $\dfrac{d\xi}{dy}=-\dfrac{d\eta}{dx}$. We may show that

$$\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}=\left\{\frac{d^2w}{d\xi^2}+\frac{d^2w}{d\eta^2}\right\}\left\{\left(\frac{d\xi}{dx}\right)^2+\left(\frac{d\xi}{dy}\right)^2\right\}.$$

Since we may interchange x, y and ξ, η in this formula, it easily follows that
$$\left\{\left(\frac{d\xi}{dx}\right)^2+\left(\frac{d\xi}{dy}\right)^2\right\}\left\{\left(\frac{dx}{d\xi}\right)^2+\left(\frac{dx}{d\eta}\right)^2\right\}=1.$$

2. Suppose we know the motion of a homogeneous membrane with given bounding conditions vibrating transversely, say $w=\phi(\xi, \eta, t)$, where w represents the displacement of a point whose coordinates are (x, y). Then this value of w satisfies
$$D_0\frac{d^2w}{dt^2}=T\left(\frac{d^2w}{d\xi^2}+\frac{d^2w}{d\eta^2}\right),$$
where D_0 is the density and T is the tension of the membrane.

Let x, y be the coordinates of a point on another membrane which has sand strewed over it and fastened to it, so that the sand vibrates with the membrane. Let the density D of this heterogeneous medium be given by
$$\frac{D}{D_0}=\left(\frac{d\xi}{dx}\right)^2+\left(\frac{d\xi}{dy}\right)^2.$$
Then the equation of motion of this new membrane is
$$D\frac{d^2w}{dt^2}=T\left(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}\right).$$
But since ξ, η are known functions of x, y, we obtain, by substitution in the equation $w=\phi(\xi, \eta, t)$, the new relation $w=\psi(x, y, t)$, which is the solution of the equation of motion of the new membrane.

Thus the motion of the new membrane is deduced from that of the first with corresponding bounding conditions.

3. Generally, we do not want the actual motion of the membrane, but only its possible periods of vibration and nodal lines. We may notice that these two membranes have the same periods of vibration and corresponding nodal lines.

4. In this transformation it is necessary that only one point of each membrane should correspond to any single point of the other membrane within the area considered. If this be not attended to, some difficulties in interpretation may occur.

5. The new membrane is of course heterogeneous, and it may be objected that the cases now considered are not such as occur in nature. If, however, the density be not very variable over the membrane, the results will nearly represent the motion of a homogeneous membrane. At the same time we must remember that the results to be obtained are not merely approximations, but are accurate solutions of the equations. Such a solution, if short, and obtained by some simple process, is sometimes preferable to one obtained by a long approximation, even though the latter may appear to be more directly applicable.

To take a simple example, the oscillations of a homogeneous loose

heavy chain, suspended from two fixed points, can be found only by very troublesome algebraical approximations. But if we suppose the chain to be heterogeneous, we may obtain an accurate solution of the equations. This solution leads to nearly the same results as the approximate investigations for a homogeneous chain. See the Author's "Rigid Dynamics."

To take another example, we may notice that the motion of a homogeneous membrane bounded by two radii vectores and two circular arcs, can be expressed by the help of Bessel's functions. But the motion of a membrane bounded in the same way and of tho proper density, can be expressed by ordinary sines and cosines. This is much simpler than a solution in Bessel's functions, and helps us to understand the nature of the motion.

6. We may, if we please, express all this in geometrical language.

Consider first a heterogeneous membrane with any fixed boundary which vibrates according to the law

$$w = \psi(x, y, t),$$

where w is the displacement of the point P whose Cartesian coordinates are x, y. Trace on the membrane the two sets of curves whose equations are $f(x, y) = \xi$ and $F(x, y) = \eta$, where ξ and η are two parameters. These curves are to be such that, when the parameters ξ, η increase by a constant increment $d\xi = a$ or $d\eta = a$, the two sets of curves divide the membrane into elementary squares. That the corresponding increments of ξ and η should be equal when these curves form squares, follows from the proposition that the small corresponding figures formed on the two membranes by the method of conjugate functions are similar. It may, however, also be deduced from the relations mentioned in Art. 1. If $ABCD$ be one of these squares, draw a parallel to the axis of x through any corner A, and then draw perpendiculars BM and DN from the two adjacent corners on this parallel. We have thus two equal triangles ABM, ADN; the sides in each triangle being the dx and dy produced by varying first ξ only, and then η only. It follows from this that $\dfrac{dx}{d\xi} = \dfrac{dy}{d\eta}$ and $\dfrac{dx}{d\eta} = -\dfrac{dy}{d\xi}$. The area of one of these squares is

$$\left(\frac{dx}{d\xi}\frac{dy}{d\eta} - \frac{dx}{d\eta}\frac{dy}{d\xi}\right) a^2.$$

Thus, since the density D is given by

$$\frac{D_0}{D} = \left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dx}{d\eta}\right)^2,$$

it follows *that the mass of each elementary square is the same.*

Next, consider the corresponding homogeneous membrane. Draw on the membrane straight lines parallel to the axes of ξ, η at a distance a

from each other, so that each straight line corresponds to one of the curves drawn on the heterogeneous membrane. Let a new boundary be drawn which cuts these straight lines at the same angles which the boundary of the heterogeneous membrane cuts the corresponding curves.

Then the motions of these two membranes are the same at corresponding points. We may consider each to be given by

$$w = \psi(x, y, t),$$

according as we express w in terms of ξ, η or x, y.

7. We may notice that the two membranes are so related that the *masses of corresponding squares on the heterogeneous and homogeneous membranes are equal to each other. Thus the whole masses of the membranes are the same, but differently distributed.*

8. Similar theorems apply in changing from one heterogeneous medium to another, but as this case does not present any novelty, and is not so simple as the one just considered, we need not discuss it minutely.

9. Having traced on the membrane the two orthogonal sets of curves $f(x, y) = \xi$, $F(x, y) = \eta$, where ξ and η are constants, and the functions both satisfy Laplace's equation, we may trace a third set of curves given

by $$\left(\frac{d\xi}{dx}\right)^2 + \left(\frac{d\xi}{dy}\right)^2 = \left(\frac{d\eta}{dx}\right)^2 + \left(\frac{d\eta}{dy}\right)^2 = \text{constant}.$$

These are, of course, the curves of constant density.

A curve of constant density which passes through any point will cut the two members of the two orthogonal sets which pass through the same point at complementary angles. *Then we may show that the sines of these angles are as the radii of curvature of the two members at that point.*

To prove this, let us find $\tan\theta$, where θ is the angle the curve of equal density makes with the curve $f(x, y) = \xi$. By simple differentiation, we find

$$\tan\theta = \frac{(f_y^2 - f_x^2) f_{xy} + 2 f_x f_y f_{xx}}{2 f_x f_y f_{xy} + (f_x^2 - f_y^2) f_{xx}},$$

where suffixes, as usual, imply differential coefficients. Since $f_x = F_y$ and $f_y = -F_x$, we see, by substituting in the numerator, that

$$\frac{\sin\theta}{\sin\theta'} = -\frac{(F_x^2 - F_y^2) F_{xx} + 2 F_y F_x F_{xy}}{2 f_x f_y f_{xy} + (f_x^2 - f_y^2) f_{xx}}.$$

But the radius of curvature ρ of the curve f is given by

$$(f_x^2 - f_y^2) f_{xx} + 2 f_x f_y f_{xy} = \frac{(f_x^2 + f_y^2)^{\frac{3}{2}}}{\rho}.$$

Hence, we see that $$\frac{\sin\theta}{\sin\theta'} = -\frac{\rho}{\rho'}.$$

10. It is not every heterogeneous medium whose motion can be deduced from that of a *homogeneous one.* If we eliminate ξ between

$$\left.\begin{array}{c}\left(\frac{d\xi}{dx}\right)^2+\left(\frac{d\xi}{dy}\right)^2=\frac{D}{D_0}\\ \frac{d^2\xi}{dx^2}+\frac{d^2\xi}{dy^2}=0\end{array}\right\},$$

we easily obtain $$\frac{d^2\log D}{dx^2}+\frac{d^2\log D}{dy^2}=0.$$

It immediately follows (from Art. 1) that

$$\frac{d^2\log D}{d\xi^2}+\frac{d^2\log D}{d\eta^2}=0.$$

The density of the heterogeneous membrane must, therefore, be such that its logarithm satisfies Laplace's equation.

11. For convenience of reference, let (x, y) be the Cartesian coordinates, (r, θ) the polar coordinates of a point P on the heterogeneous membrane; (ξ, η) the Cartesian, (ρ, ω) the polar coordinates of the corresponding point Π on the homogeneous membrane.

Suppose we take as our relation between the two points

$$\xi+\eta\sqrt{-1}=c\log\frac{x+y\sqrt{-1}}{\beta}.$$

Then we find $$\left.\begin{array}{l}\xi=c\log\frac{r}{\beta}\\ \eta=c\theta\end{array}\right\}.$$

Thus straight boundaries on the homogeneous membrane parallel to the axis of ξ correspond to straight boundaries on the heterogeneous membrane which pass through the origin. At the same time, straight boundaries parallel to the axis of η correspond to circles whose centre is at the origin.

The density D is given by

$$\frac{D}{D_0}=\left(\frac{d\xi}{dr}\right)^2+\left(\frac{d\xi}{rd\theta}\right)^2=\left(\frac{c}{r}\right)^2.$$

If r vanish, we have D infinite; it will therefore be necessary to exclude the origin from the area of the membrane.

12. *If, then, we know the motion of a membrane bounded by a rectangle, this transformation immediately gives the motion of a heterogeneous membrane bounded by two circular arcs and any two radii vectores.*

Example.—The motion of a rectilinear homogeneous membrane bounded by the straight lines $\xi=h_1$, $\xi=h_2$; $\eta=k_1$, $\eta=k_2$, is known to be given by the type

$$w=A\sin m\pi\frac{\xi-h_1}{h_2-h_1}\sin n\pi\frac{\eta-k_1}{k_2-k_1}\cos pt,$$

where the integers m, n are any which satisfy

$$\frac{m^2}{(h_2-h_1)^2}+\frac{n^2}{(k_2-k_1)^2}=\frac{p^2}{a^2\pi^2},$$

where $$a^2 = \frac{T_0}{D_0}.$$

It immediately follows that the motion of a heterogeneous membrane bounded by the arcs of concentric circles, whose radii are h_1' and h_2', and by two radii vectores $\theta = \alpha_1$ and $\theta = \alpha_2$, is given by

$$w = A \sin\left(m\pi \frac{\log r - \log h_1'}{\log h_2' - \log h_1'}\right) \sin\left(n\pi \frac{\theta - \alpha_1}{\alpha_2 - \alpha_1}\right) \cos pt,$$

where the integers m and n are connected by the equation

$$\frac{m^2}{(\log h_2' - \log h_1')^2} + \frac{n^2}{(\alpha_2 - \alpha_1)^2} = \frac{c^2p^2}{a^2\pi^2},$$

and the density D of the membrane is given by

$$\frac{D}{D_0} = \left(\frac{c}{r}\right)^2.$$

13. Another useful relation between the corresponding points P and Π is $$\xi + \eta\sqrt{-1} = c\left(\frac{x + y\sqrt{-1}}{c}\right)^n.$$

This gives $$\left.\begin{aligned} \xi &= c\left(\frac{r}{c}\right)^n \cos n\theta \\ \eta &= c\left(\frac{r}{c}\right)^n \sin n\theta \end{aligned}\right\};$$

and therefore, in polar coordinates,

$$\left.\begin{aligned} \rho &= c\left(\frac{r}{c}\right)^n \\ \omega &= n\theta \end{aligned}\right\}.$$

By this transformation all radii vectores are turned round the origin and altered in a known manner.

Also, the density D of the heterogeneous membrane is given by

$$\frac{D}{D_0} = n^2\left(\frac{r}{c}\right)^{2(n-1)}.$$

Since $\theta =$ constant makes $\omega =$ a constant, we see that straight lines through the origin correspond to straight lines through the origin. Also, circles whose centres are at the origin correspond to circles whose centres are at the origin.

If we choose $n = -1$, we have the ordinary case of inversion; thus

$$\left.\begin{aligned} \rho &= \frac{c^2}{r} \\ \omega &= -\theta \end{aligned}\right\}.$$

In this case any circle inverts into a circle. The density of the membrane is then given by $\frac{D}{D_0} = \left(\frac{c}{\rho}\right)^4$. As this is infinite when ρ is zero, the centre of inversion must be external to the membrane.

14. *Example.*—The density of a membrane bounded by two concentric fixed circles of radii a and b at any point distant ρ from the centre is $\frac{A}{\rho^2}$. Let it vibrate symmetrically so that the nodal lines are concentric circles, then it may easily be proved that the periods of vibration are $\frac{2\pi}{p}\left(\frac{A}{T}\right)^{\frac{1}{2}}$, where p is such that $p\,(\log a - \log b)$ is a multiple of π.

Let us invert this with regard to an external point. We immediately have this theorem.

A heterogeneous membrane is bounded by two fixed circles, centres C and C'. Let O be that point which has a common polar line in both circles, and let this polar line cut the straight line OCC' in the point R. Let the density of this membrane at any point P be given by

$$D = A \,.\left(\frac{OR}{OP\,.\,RP}\right)^2.$$

Then this membrane can vibrate so that the nodal lines are circles, and the possible periods of vibration are $\frac{2\pi}{p}\left(\frac{A}{T}\right)^{\frac{1}{2}}$, where p is such that

$$p \log \frac{a\,.\,OC'}{a'\,.\,OC} = \text{multiple of } \pi,$$

and where a and a' are the radii of the circles whose centres are C and C'.

15. *Example.*—The motion of a rectilinear membrane bounded by the axes of ξ and η and the straight lines $\xi = h$, $\eta = k$, is known to be given by the type
$$w = A \sin \frac{m\pi\xi}{h} \sin \frac{n\pi\eta}{k} \cos pt,$$
where the integers m and n are any which satisfy

$$\frac{m^2}{h^2} + \frac{n^2}{k^2} = \frac{p^2}{a^2\pi^2}.$$

Let us invert this with regard to the origin, we see that—

The motion of an infinite membrane bounded by the axes of x and y, and the arcs of two circles whose radii are h', k', and which touch the axes of x, y at the origin, is given by the type

$$w = A \sin \frac{m\pi h' \cos\theta}{r} \sin \frac{n\pi k' \sin\theta}{r} \cos pt,$$

where the integers m and n satisfy the equation

$$m^2h'^2 + n^2k'^2 = \frac{p^2}{a^2\pi^2}\, c^4,$$

provided its density is given by

$$D = \left(\frac{c}{r}\right)^4 . \frac{T_0}{a^2},$$

where T_0 = tension of the membrane.

16. *Example.*—If we transform the same theorem with $n = 2$, we see that—

The motion of a finite membrane bounded by two straight lines $OA = h'$, $OB = k'$, inclined at an angle $\frac{\pi}{4}$, and by two rectangular hyperbolas passing respectively through A and B, and having OB and OA for asymptotes, is given by the type

$$w = A \sin \frac{m\pi r^2 \cos 2\theta}{h'^2} \sin \frac{n\pi r^2 \sin 2\theta}{k'^2} \cos pt,$$

where m and n are connected by

$$\frac{m^2}{h'^2} + \frac{n^2}{k'^2} = \frac{p^2}{a^2\pi^2} \frac{1}{c^2},$$

provided its density is given by

$$D = 4\left(\frac{r}{c}\right)^2 \cdot \frac{T_0}{a^2}.$$

17. Suppose, in an infinite homogeneous membrane, a very small circular area of radius c to become rigid, and to be constrained to move transversely with a motion given by $w = A \cos apt$. Then waves will spread out equally in all directions, and when the motion has become steady, the vibration at any point distant ρ from the centre of disturbance is given by $$w = J_0(p\rho) A \cos apt.$$
Here we have supposed c to be so small that $J_0(pc) = 1$. Such a small circular vibrating area may, for convenience, be called *a source of disturbance*, or more shortly *a source.*

If we transform this theorem by the method of conjugate functions, we see, for the reason to be given in Art. 22, that the infinitely small circle will transform into a similar figure, *i.e.*, into another circle.

18. *Example.*—The vibrations of an infinite homogeneous membrane bounded by a fixed straight line taken as the axis of x, and acted on by a *source* at some point (ξ_1, η_1), are given by

$$w = \{J_0(p\rho) - J_0(p\rho')\} A \cos apt,$$

where $$\rho^2 = (\xi - \xi_1)^2 + (\eta - \eta_1)^2,$$

and $$\rho'^2 = (\xi - \xi_1)^2 + (\eta + \eta_1)^2,$$

so that ρ, ρ' are the distances of the point (ξ, η) from the source, and its image on the other side of the axis of ξ.

Hence we infer that the vibrations of an infinite heterogeneous membrane bounded by two fixed radii vectores forming a corner of angle $\frac{\pi}{n}$, and acted on by a source at a point $r_1\theta_1$, are given by

$$w = \{J_0(pR) - J_0(pR')\} A \cos apt,$$

where $$c^{2n-2}R^2 = r^{2n}+r_1^{2n}-2r^n r_1^n \cos n\,(\theta-\theta_1),$$
$$c^{2n-2}R'^2 = r^{2n}+r_1^{2n}-2r^n r_1^n \cos n\,(\theta+\theta_1),$$
provided the density of the membrane is given by
$$\frac{D}{D_0} = n^2\left(\frac{r}{c}\right)^{2(n-1)}.$$
Here r, θ are the running coordinates of any point of the medium, and w is the tranverse displacement at the point $\rho\omega$, and D_0 is a constant.

Part II.

19. The method of conjugate functions has been much used in solving Hydro-dynamical problems, and I therefore do not suppose that the following remarks can contain anything which is perfectly new. But, in all the applications which I have seen, the change from one motion to another of a simpler character has been effected by the use of the velocity and stream potentials. Frequently these are very complicated, and besides they have to be differentiated and discussed, in order to find the actual motion of the fluid. It seems to me that, when we thoroughly know one motion, we ought to be able to state, without any long discussion, exactly what occurs in any derived motion. I propose, therefore, to state a few elementary propositions, by which I think this may be effected, and to illustrate them by some easy examples which admit of solution in finite algebraical terms.

20. Let us imagine two areas to be occupied, each by a fluid in motion. Let ξ, η be the coordinates of a point Π in one, and x, y the coordinates of a corresponding point P in the other, so related that
$$\xi+\eta\sqrt{-1} = f(x+y\sqrt{-1}).$$
Thus, when the function f has been chosen, ξ, η are known functions of x, y.

Let ϕ and ψ be the velocity and current function of any motion within the first area, given by
$$\phi+\psi\sqrt{-1} = \chi_1(\xi+\eta\sqrt{-1}),$$
and let the boundary be $F_1(\xi, \eta) = 0$. If we substitute for ξ, η their values in terms of x, y, suppose, we have
$$\phi+\psi\sqrt{-1} = \chi_2(x+y\sqrt{-1}),$$
$$F_1(\xi, \eta) = F_2(x, y).$$
Then these same functions ϕ and ψ will now be the velocity and current functions of a motion in the second area with a boundary $F_2(x, y) = 0$.

Thus a motion in the second area, corresponding to a given motion in the first area, has been found by help of the velocity and current functions. From this we may deduce the three following propositions.

21. Prop. 1.—It is obvious that ξ, η are themselves the velocity a.
current functions of some motion; if then we write

$$\mu^2 = \left(\frac{d\xi}{dx}\right)^2 + \left(\frac{d\xi}{dy}\right)^2 = \left(\frac{d\eta}{dx}\right)^2 + \left(\frac{d\eta}{dy}\right)^2,$$

we may call μ the velocity of the transformation. Thus μ^2 is what has been called $\frac{D}{D_0}$ in treating of membranes. See Art. 2.

We may then deduce from Art. 20, that if Π and P be corresponding particles of fluid, $\begin{pmatrix}\text{vel. of}\\ P\end{pmatrix} = \begin{pmatrix}\text{vel. of}\\ \Pi\end{pmatrix} \times \begin{pmatrix}\text{velocity of}\\ \text{transformation}\end{pmatrix}$.

Thus the *actual velocities* at any two corresponding points may be compared. The *directions of motion* at corresponding points are such that they make equal angles with corresponding lines in the two areas.

If we trace out any two elementary areas in the two fluids by means of corresponding points, we see, by Art. 6, that μ^2 expresses the ratio of these areas. It immediately follows, that *the kinetic energies of the two fluids which occupy corresponding areas are equal. Thus the whole kinetic energies of the two motions are equal, but differently distributed over the areas of motion.*

This proposition should be compared with the corresponding proposition in the motion of membranes given in Art. 7.

22. Prop. 2.—*If "a source" or "a vortex" exist in one fluid, there will be "a source" or "a vortex" at the corresponding point of the other fluid. These will be of equal strength in the two motions.* This follows partly from the last proposition, and partly from the fact that the angle between any two curves on one area is equal to the angle between corresponding curves on the other area. Thus, if any infinitely small network of curves be drawn on one area, and the corresponding network on the other, these two networks are similar to each other.

If we sketch, as in Art. 6, the curves whose parameters are ξ and η, it is assumed that these curves have no double or multiple point at the source or vortex, so that the differential coefficients of ξ and η with regard to x and y are finite. If the circumstances of the problem permit the existence of a point of multiplicity of degree n, then *a source or a vortex at the multiple point will transform into a source or vortex of n times its strength.*

23. Prop. 3.—Suppose a vortex Π of strength m to exist in the firs' fluid, at a point whose coordinates are ξ, η. Then there will be vortex P of equal strength at the corresponding point (x, y) of the secon fluid. But these do not necessarily continue to move so as to occupy corresponding points. We may, however, infer the motion of P from that of Π by the following rule:—

Let $\chi(\xi, \eta)$ *be a current function* (not the current function of the fluid) *giving the motion of the vortex* Π, *so that its velocities* (not the velocities of the fluid) *resolved parallel to the axes of* ξ *and* η *are respectively* $\frac{d\chi}{d\eta}$ *and* $-\frac{d\chi}{d\xi}$. *Then the motion of* P *is given by a current function* $\chi'(x, y) = \chi(\xi, \eta) - \frac{m}{2}\log\mu$, *i.e., its velocities resolved parallel to the axes of* x, y *are respectively* $\frac{d\chi'}{dy}$ *and* $-\frac{d\chi'}{dx}$, *and its path is found by equating* χ' *to a constant. Here* μ *is the velocity of the transformation as defined in Art.* 21.

In this proposition, ξ, η are regarded, as before, as known functions of x, y, such that their differential coefficients with regard to x, y are finite.

Of course we may easily change this enunciation so as to avoid the use of the function χ. And we may use any kind of coordinates. Generally, we may say, *the current function of* P *is obtained from that of* Π *by subtracting* $\frac{m}{2}\log\mu$.

24. The method of conjugate functions seems to me a much more convenient method of solving a certain class of Hydro-dynamical problems than the method of Images. By a proper choice of the formulæ of transformation, we may deduce the motion with a complicated boundary from that with some simple boundary. Frequently, instead of a long series of images, whose effects we have to sum, we have merely to make some simple substitution. On the other hand, there is the difficulty of finding the proper formulæ of transformation by which to simplify the given boundary. But these we may frequently obtain by the following rule :—

25. Suppose we require the motion of a fluid with sources or vortices P_1, P_2, &c., within an infinite area whose boundary is given by $F_2(x, y) = 0$. Let us remove these sources or vortices, and try to find a steady a-cyclic motion of fluid with the same boundary. Suppose this can be done, and let ξ, η be the velocity and stream potentials, so that η is constant along the boundary F_2; let this constant value be $\eta = k$. Then, if we use ξ, η as our formulæ of transformation, the given boundary F_2 will transform into a straight line $\eta = k$, and the area of the motion will in general transform into the infinite space on one side of this straight line. Now replace the sources and vortices P_1, P_2, &c., there will be corresponding sources and vortices Π_1, Π_2, &c. in the space bounded by the straight line $\eta = k$. The motion of these may, in general, be found by placing a solitary image for each on the other side of the straight line. Hence, the motion of P_1, P_2, &c. may be at once inferred by the rule given in Art. 23.

26. This method of finding the proper formulæ of transformation applies conveniently when the given boundary is formed by a single curve extending to infinity in both directions, which can thus be unwrapped into the straight line $\eta = k$. If the boundary be formed by two non-intersecting curves, each of which extends to infinity, it is possible the constant values of η may be different along the two curves. Let these values be k, k', in this case the motion simplifies into that between the two parallel straight lines $\eta = k$, $\eta = k'$. If there be three such boundaries, the motion simplifies into that between three parallel straight lines which are not all infinite in both directions.

If the boundary be finite it will be more convenient to regard ξ, η as polar coordinates. If we put $\xi = \omega$, $\eta = \log \rho$, where (ρ, ω) are polar coordinates of a point Π corresponding to P, we may still regard ξ, η as conjugate functions of x, y. In this case the finite boundary will transform into a circle. The motion of a vortex outside a circle being known, that of a vortex outside the given boundary may be deduced. See Art. 29.

The general result is, that *if we know an a-cyclic motion of a fluid within the space bounded by one or two infinite curves, or the cyclic motion of a fluid round the outside of a finite boundary, we can, in general, find the motion with the same boundary when complicated by the presence of sources and vortices.*

The relations which exist between the conditions at infinity in the two motions, may be deduced from the formulæ of transformation, or from the rules given in Art. 21. In general, since the whole energy is unaltered, if the fluid be at rest at infinity in one motion, it will be at rest at infinity in the other.

27. *Example* 1.—Let it be required to find the motion of a fluid with a vortex P_1, moving in a corner bounded by two straight lines inclined at an angle α. This is Prof. Greenhill's problem, see "Quarterly Journal," Vol. xv.

Removing the vortex, it is well known that a possible motion in such a corner is given by the velocity and current potentials ξ, η, where ξ, η have the values given in Art. 13, and $n\alpha = \pi$. Also, looking at these values, we see that the area in the corner transforms into the whole space on one side of the axis of ξ. These, then, are the proper formulæ of transformation.

The motion of a fluid bounded by the axis of ξ with a vortex Π_1 at the point (ξ_1, η_1) is given by the current function

$$\psi = -\frac{m}{2} \log \{(\xi-\xi_1)^2 + (\eta-\eta_1)^2\} + \frac{m}{2} \log \{(\xi-\xi_1)^2 + (\eta+\eta_1)^2\},$$

which is obtained by placing an image of the vortex at the point $(\xi_1, -\eta_1)$ and super-imposing the two motions.

Taking any point in the axis of ξ as origin, we shall turn the negative side of the axis round the origin until it makes an angle equal to a with the positive side. To effect this we express ξ, η in polar coordinates (ρ, ω), and then write $\rho = c\left(\frac{r}{c}\right)^n$ and $\omega = n\theta$. Thus the stream function at a point (r, θ) for fluid moving in a corner with a vortex P_1 at (r_1, θ_1) is

$$\psi = -\frac{m}{2}\log\{r^{2n}+r_1^{2n}-2r^n r_1^n \cos n(\theta-\theta_1)\}$$
$$+\frac{m}{2}\log\{r^{2n}+r_1^{2n}-2r^n r_1^n \cos n(\theta+\theta_1\}.$$

The motion of the vortex itself may be deduced after some algebraic reductions from this value of ψ. But the rule in Art. 23 enables us to write down the result. The vortex Π_1 evidently moves parallel to the axis of ξ with a velocity $\frac{m}{2\eta_1}$; hence its stream function is $\frac{m}{2}\log\eta_1$. The velocity of transformation is $\mu = n\left(\frac{r}{c}\right)^{n-1}$. The stream function which gives the motion of the vortex in the corner is therefore

$$\chi' = \frac{m}{2}\log(\rho_1 \sin\omega_1) - \frac{m}{2}\log\mu$$
$$= \frac{m}{2}\log(r_1 \sin n\theta_1).$$

The path is therefore given by $r_1 \sin n\theta_1 =$ a constant. It has not been assumed in this investigation that n is an integer. If n be less than unity, the angle of the corner exceeds two right angles; in this case we cannot assert without investigation that the pressure would be positive near the corner.

If two circles intersect in A and B, we may find, by inverting this result, the *motion of a vortex V in the space between the circular boundaries.* Let θ be the angle the circle through A, B and the vortex V makes with either circular boundary, and let a be the angle between the circular boundaries. Then the current function of the vortex V is found by subtracting $\frac{m}{2}\log\mu$ from the value of χ' given above, where $\mu = \left(\frac{c}{r}\right)^2$, as shown in Arts. 13 and 21. The current function of the vortex V is therefore

$$\chi = \frac{m}{2}\log\left(AV.BV.\sin\frac{\pi\theta}{a}\right).$$

The path of the vortex is given by the equation

$$AV.BV.\sin\frac{\pi\theta}{a} = \text{constant},$$

which can be traced without difficulty.

28. *Example* 2.—When the given boundaries are straight lines parallel to the axes, we may obtain the proper formulæ of transformation, by solving the equation

$$\frac{d^2\eta}{dx^2}+\frac{d^2\eta}{dy^2}=0,$$

by the method usually adopted in linear equations. We have $\eta=\Sigma Ae^{px}\sin qy$, where $p^2=q^2$.

Taking $\eta=e^{px}\sin py$, this is the stream function of a motion within the space bounded by the straight lines $y=0$, $py=\pi$, and closed at the infinitely distant extremity $x=-\infty$. This gives $\xi=e^{px}\cos py$, since $\frac{d\xi}{dx}=\frac{d\eta}{dy}$. The space being closed at one extremity, we have, in reality, but a single boundary which we unwrap into the axis of ξ. To find the motion of a vortex in such a space, we first find its motion in the infinite space bounded by the axis of ξ. If this be expressed in polar coordinates (ρ, ω), we write in our expression $\rho=e^{px}$ and $\omega=py$. Corresponding lines at Π and P are the radius vector and a parallel to the axis of x; the ratio of the velocities at Π and P is pe^{px}.

If we wish the space between the given straight lines to be open at both ends, we merely super-impose on every particle a velocity equal and opposite to half that of the velocity at the extremity $x=+\infty$.

If we take $\eta=(e^{px}-e^{-px})\sin py$, we have the space bounded by the axis of y, and two straight lines parallel to the axis of x, viz., $y=0$ and $py=\pi$. This gives $\xi=(e^{px}+e^{-px})\cos py$. These formulæ of transformation enable us to find a motion with a source or vortex in such a space.

To take a special case, we notice that the stream function of a vortex moving in the space bounded by the axis of ξ is $\frac{m}{2}\log\eta$, as explained in Art. 27. Hence, the stream function of a vortex in the space between the axis of x and a parallel to this axis at a distance h, is $\frac{m}{2}\log\eta-\frac{m}{2}\log pe^{px}$. Since $\eta=e^{px}\sin py$, this reduces to

$$\chi=\frac{m}{2}\log\sin\frac{\pi y}{h},$$

after omitting an unnecessary constant.

Inverting this last result, we see that the stream function of a vortex in the space between two circles touching at a point O, is

$$\chi=\frac{m}{2}\log r^2\sin\frac{\pi c'}{c-c'}\left(\frac{c}{r}\cos\theta-1\right),$$

where c, c' are the lengths of the diameters, O is the origin, and θ is measured from the axis of symmetry.

29. *Example* 3.—What formulæ of transformation should we use to find the motion of a fluid with vortices and sources outside a fixed curve of any form, so as to simplify the motion to the corresponding motion about a circle?

Consider the curve for a moment as a section of an infinite cylinder, charged with any given quantity of electricity. Let η be the potential at any external point P, whose coordinates are x, y. Then η is constant along the given curve, say $\eta = \eta_1$. Also the differential coefficients of η will be finite if the given curve (whatever it be) have no cusps pointing outwards. Take another function ξ, so that

$$\xi = \int \left(\frac{d\eta}{dy}\, dx - \frac{d\eta}{dx}\, dy \right),$$

then ξ and η are conjugate functions of x and y. In a Hydro-dynamic point of view, ξ and η are the velocity and current functions of the motion of a fluid round the given closed boundary, and therefore, by Art. 25, supply a proper set of formulæ for transformation.

If we now put $\xi = \omega + A$ and $\eta = \log \rho + B$, where A and B are some constants to be chosen presently, we see, by Art. 11, that (ρ, ω) are the polar coordinates of a point Π which may be derived from the point P whose Cartesian coordinates are (x, y) by the method of conjugate functions. The given boundary which is defined by the constant value of η, viz., $\eta = \eta_1$, will then transform into a circle whose radius is given by the value of ρ corresponding to $\eta = \eta_1$.

Since η is the potential of a distribution of matter on a curve of finite dimensions, it is clear that, when P is at infinity, η will take the form $\eta = E \log r + \beta$, and therefore ξ will take the form $\xi = E\theta + \alpha$, where E depends on the mass of the distributed matter, and may be taken unity, (α, β) are two constants depending on the form of the curve, and (r, θ) are the polar coordinates of P. Comparing these with the expressions for ξ and η in terms of the coordinates (ρ, ω) of Π, we see that, if we take $A = \alpha$, $B = \beta$, and $E = 1$, the two sets of expressions will become the same. Thus, whatever conditions at infinity hold in one motion, the same conditions will hold in the transformed motion.

Any motion, therefore, about a closed curve, for which η is known, may in general be deduced from a corresponding motion about a circle.

In this transformation, corresponding lines at P and Π are the lines of force and the radii vectores. The ratio of the velocity at P to that at Π is $\dfrac{F}{\rho}$, where F is the force at P.

30. If, for example, the given curve be an ellipse whose semi-axes are a_1, b_1, the level curves are known to be confocal ellipses, and the

lines of force confocal hyperbolas. It has just been proved that the value of ξ for any branch of a line of force which lies outside the given curve, is some multiple of the angle the asymptote to *that* branch makes with a straight line fixed in space, say the major axis. Hence if (a', b') be the semi-axes of any confocal hyperbola, we have

$$\xi = \tan^{-1}\frac{b'}{a'}.$$

Again, the ellipse and hyperbola may be derived from each other by the same analytical process, hence the value of η at any external point is some multiple of the angle the imaginary asymptote makes with a fixed straight line. If, then, (a, b) be the semi-axes of any confocal ellipse, we have $\eta = c\tan^{-1}\frac{b\sqrt{-1}}{a}$. Putting unity for $c\sqrt{-1}$, and omitting the constant $-\log h$, where $h^2 = a^2 - b^2$, this reduces to $\eta = \log(a+b)$. We may, however, avoid the use of imaginary quantities. Let dn be an element measured outwards, of a normal to a level curve at any point P, ρ the radius of curvature at P, then $\log\frac{d\eta}{dn} = -\int\frac{dn}{\rho}$. Taking dn along the minor axis, we have $dn = db$ and $\rho = \frac{a^2}{b}$. A very simple integration then gives η.

We have $\xi = \tan^{-1}\frac{b'}{a'} = \sin^{-1}\frac{b'}{h}$, where $h^2 = a'^2 + b'^2$. We have also supposed that ξ varies from 0 to 2π as P travels round the ellipse. If, then, we consider h to be always positive, we see that a' must follow the sign of the *abscissa* of P, and b' must follow the sign of the *ordinate* of P.

31. If fluid moving at infinity with a velocity V parallel to the axis of x, from $+\infty$ towards $-\infty$, flow round a circular disc of radius c, we know that the motion at any point (ρ, ω) is given by

$$\phi + \psi\sqrt{-1} = -V\left(\rho e^{\omega\sqrt{(-1)}} + \frac{c^2}{\rho}e^{-\omega\sqrt{(-1)}}\right),$$

so that the current function ψ is given by

$$\psi = -V\left(\rho - \frac{c^2}{\rho}\right)\sin\omega.$$

Let us find the corresponding motion round an ellipse whose semi-axes are (a_1, b_1), placed with its major axis in the direction of motion.

Taking the two values of ξ, η already found, we have

$$\left.\begin{aligned}\xi &= \omega + A = \tan^{-1}\frac{b'}{a'}\\ \eta &= \log\rho + B = \log(a+b)\end{aligned}\right\}.$$

To make the conditions at infinity the same, we have $A = 0$, $B = \log 2$.

Hence $\rho = \frac{1}{2}(a+b)$. Also, at the boundary of the given ellipse, we have $c = \frac{1}{2}(a_1+b_1)$. The current function for the motion about the ellipse is therefore $\psi = -\frac{1}{2}V\left(a+b-\frac{(a_1+b_1)^2}{a+b}\right)\frac{b'}{h}$.

Here (a, b) are the elliptic coordinates of P, so that the Cartesian coordinates (x, y) of P are

$$x = \frac{aa'}{h},\ y = \frac{bb'}{h},\ \text{and}\ h^2 = a^2-b^2 = a'^2+b'^2.$$

We may also write this current function in the form

$$\psi = -V\left(b-\frac{(a_1+b_1)\,b_1}{a+b}\right)\frac{b'}{h}.$$

Note on Abel's Theorem. *By* THOMAS CRAIG, United States Coast and Geodetic Survey.

[*Read February* 10*th*, 1881.]

Denoting by s and z two variables, let

$$F(s, z) = 0$$

express an algebraic relation connecting them and one which is incapable of reduction to any simpler form. The solution of this equation gives s as a many-valued function of z, and one for the single-valued spread of which we require a Riemann's, say $2p+1$-fold, surface, which we may denote by R. Denoting then by f a rational function of s and z, and considering, as usual, only discontinuities of a polar nature, let

$$p_1, p_2 \cdots p_n$$

denote the poles of f, and $\quad q_1, q_2 \cdots q_n$

the zero points of f. Draw, in the usual manner, the systems of cuts ordinarily denoted by a, b, which have the effect of reducing the surface R to a simply-connected surface, say R_1. Draw also the non-intersecting lines

$$p_1 \cdots\cdots q_1,$$
$$p_2 \cdots\cdots q_2,$$
$$\cdots \quad \cdots \quad \cdots$$
$$p_n \cdots\cdots q_n,$$

upon R_1. If we should conceive R_1 cut through along these lines (pq) we should obviously have a new surface, part of whose boundaries would be the two sides of these cuts; without actually cutting the surface, however, we may still consider the lines (pq) as the partial boundaries of the new surface, which we may denote by R_{11}. With the ordinary conventions for the positive bounding of a surface, we may

ay at once, without entering into elementary explanation, that whatever value $\log f$ may have upon the negative side of one of the lines (pq), its value on the positive side will be greater than on the negative by $2\pi i$. Also, the value of $\log f$ on the positive side of a will be greater than on the negative side by $2m\pi i$, and on the positive of b greater than on the negative side by $2n\pi i$; m and n being integers which in general depend upon the form of the function f and the position of the cuts a, b. Let z_k denote a particular point on the surface R where, in general, g sheets hang together. Calling the indefinitely small area immediately surrounding a point the *region* (Gebiet) of that point, we know that the region of any point z_k is formed by a line which goes round the point g times ($g = 1, 2, 3 \ldots p$). In order to prove the theorem in view, it is necessary to find a function, say w, which will give the integral

$$\int \log f \, dw = 0$$

over the boundary of a modification of the surface R_{11}; *i.e.*, we must find a function w such that

$$\log f \frac{dw}{dz}$$

shall be single-valued over the modified form of R_{11}. Denoting then by

$$\frac{dw}{dz}$$

a rational function of s and z, we can develop this function in the region of a point z_k (the points z_k are supposed not to coincide with the points p, q) in ascending powers of $(z-z_k)^{\frac{1}{g}}$, *i.e.*,

$$\frac{dw}{dz} = (z-z_k)^{\frac{\mu}{g}} \{E_\mu + (z-z_k) E_{\mu+1} + (z-z_k)^2 E_{\mu+2} + \ldots\}.$$

For the poles of $\frac{dw}{dz}$ we need only consider the negative values of μ, that is, we need this development only in the regions of points z_k where μ is negative. For an infinitely distant point we, of course, have

$$\frac{dw}{dz} = z^{-\frac{\nu}{g}} \{E'_\nu + z^{-1}E'_{\nu+1} + z^{-2}E'_{\nu+2} + \ldots\}.$$

Around each point z_k draw a small circle on R_{11}, which denote by c_k; c_k in general goes round the point c_k g times. Similarly, draw a circle c_t around the origin as centre, enclosing *all* the points z_k, and excluding points z_∞; let now R_{11} be cut through along these circles, and a new surface R_{111} will be formed, which is the modification of R_{11} referred to above. The boundaries of R_{111} are formed by the cuts a, b, the lines (pq), and the circular cuts c_k and c_t. The function

$$\frac{dw}{dz} \log f$$

is now, as we know by the elementary properties of the theory of functions, a single-valued and continuous function all over R_{111}, and we therefore have

$$\int_{R_{111}} \log f\, dw = 0.$$

Denoting now by A and B the moduli of periodicity of the function w at the cuts a and b, we have, by separation of this integral into its several parts,

$$2\pi i \sum_i \int_{p_i}^{q_i} dw + 2\pi i \sum_j n_j B_j - 2\pi i \sum_j m_j A_j + \sum_k \int_{c_k} \log f\, dw + \sum_t \int_{c_t} \log f\, dw = 0.$$

For the last two integrals we need only, as before, to develop $\log f$ in the region of the points z_k; thus

$$\log f = G_0 + G_1 (z - z_k)^{\frac{1}{g}} + G_2 (z - z_k)^{\frac{2}{g}} + \ldots,$$

and outside the circle c_t

$$\log f = G'_0 + G'_1 z^{-\frac{1}{g}} + G'_2 z^{-\frac{2}{g}} + \ldots.$$

For the quantities E, G, E', G', we have obviously

$$E_n = \frac{1}{\underline{n-\mu|}} \frac{d^{n-\mu}}{d\zeta^{n-\mu}} \left[\zeta^{-\mu} \frac{dw}{dz}\right] \begin{matrix} z = z_k \\ s = s_k \end{matrix}$$

$$E'_n = \frac{1}{\underline{n-\nu|}} \frac{d^{n-\nu}}{d\eta^{n-\nu}} \left[\eta^{-\nu} \frac{dw}{dz}\right] \begin{matrix} z = \infty \\ s = s_t \end{matrix}$$

in which

$$\zeta = (z - z_k)^{\frac{1}{g}},$$

$$\eta = z^{-\frac{1}{g}},$$

s_t being the value of s for $z = \infty$. Also, we have

$$G_n = \frac{1}{\underline{n|}} \left[\frac{d^n \log f}{d\zeta^n}\right] \begin{matrix} z = z_k \\ s = s_k \end{matrix}$$

$$G'_n = \frac{1}{\underline{n|}} \left[\frac{d^n \log f}{d\eta^n}\right] \begin{matrix} z = \infty. \\ s = s_t \end{matrix}$$

Substitution of these in the values of $\log f$, given above, conducts in a very simple manner to

$$\int_{c_k} \log f\, dw = 2\pi i g \{G_0 E_{-g} + G_1 E_{-g-1} + \ldots\},$$

$$\int_{c_t} \log f\, dw = 2\pi i g \{G'_0 E'_g + G'_1 E'_{g-1} + \ldots\}.$$

We have, therefore, the required theorem in the form

$$\sum_i \int_{p_i}^{q_i} dw + \sum_j m_j B_j - \sum_j n_j A_j$$
$$+ \sum_k g \{ G_0 E_{-g} + G_1 E_{-g-1} + \dots G_{-\mu-g} E_\mu \}$$
$$+ \Sigma g \{ G'_0 E_g + G'_1 E_{g-1} + \dots G'_{g-\nu} E'_\nu \} = 0,$$

$2\pi i$ being a common factor, and so disappearing.

On some Definite Integrals expressible in terms of the First Complete Elliptic Integral and of Gamma Functions. By J. W. L. Glaisher, M.A., F.R.S.

[*Read February* 10*th*, 1881.]

1. *Theorem.*—If the value of the integral

$$\int_0^\infty \phi(t^2)\,dt$$

be denoted by A, then

$$\int_0^\infty \int_0^\infty \phi(x^4 + 2x^2y^2 \cos 2\gamma + y^4)\,dx\,dy = \tfrac{1}{2}A \,.\, F^1(\sin\gamma)\,;$$

where $F^1(\sin\alpha)$ denotes the complete elliptic integral of the first kind whose modulus is $\sin\gamma$.

To prove this, substitute $\sqrt{c}\,.\,t$ for t in the equation

$$A = \int_0^\infty \phi(t^2)\,dt,$$

and we have
$$\frac{A}{\sqrt{c}} = \int_0^\infty \phi(ct^2)\,dt\,;$$

whence, putting $c = 1 - k^2 \sin^2\theta$,

$$\frac{A}{\sqrt{(1-k^2\sin^2\theta)}} = \int_0^\infty \phi(t^2 - k^2t^2 \sin^2\theta)\,dt\,;$$

and therefore, integrating with regard to θ between the limits 0 and π,

$$A\int_0^\pi \frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}} = \int_0^\infty \int_0^\pi \phi(t^2 - k^2t^2\sin^2\theta)\,dt\,d\theta$$
$$= \int_0^\infty \int_0^\pi \phi(t^2 - 4k^2t^2 \sin^2\tfrac{1}{2}\theta \cos^2\tfrac{1}{2}\theta)\,dt\,d\theta.$$

Transforming the double integral by putting $r^2 = t$ and replacing $\frac{1}{2}\theta$ by

θ, it becomes $\quad = 4\int_0^\infty \int_0^{\frac{1}{2}\pi} \phi\,(r^4 - 4k^2 r^4 \sin^2\theta \cos^2\theta)\, r\,dr\,d\theta\,;$

which, transforming from polar to rectangular coordinates,

$$= 4\int_0^\infty \int_0^\infty \phi\,(x^4 + 2x^2y^2 + y^4 - 4k^2x^2y^2)\,dx\,dy.$$

Now
$$\int_0^\pi \frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}} = 2F^1(k),$$

and if $k = \sin\gamma$, $\qquad 1-2k^2 = \cos 2\gamma$,

so that it has been shown that

$$A\,.\,2F^1(\sin\gamma) = 4\int_0^\infty \int_0^\infty \phi\,(x^4 + 2x^2y^2\cos 2\gamma + y^4)\,dx\,dy,$$

which is the result ennunciated above.

2. As particular cases, since

$$\int_0^\infty e^{-t^2}\,dt = \tfrac{1}{2}\sqrt{\pi},$$
$$\int_0^\infty e^{-\sqrt{t^2}}\,dt = 1,$$
$$\int_0^\infty \frac{dt}{1+t^2} = \tfrac{1}{2}\pi,$$

we have
$$\int_0^\infty \int_0^\infty e^{-(x^4+2x^2y^2\cos 2\gamma+y^4)}\,dx\,dy = \tfrac{1}{4}\sqrt{\pi}\,.\,F^1(\sin\gamma),$$
$$\int_0^\infty \int_0^\infty e^{-\sqrt{(x^4+2x^2y^2\cos 2\gamma+y^4)}}\,dx\,dy = \tfrac{1}{2}\quad.\,F^1(\sin\gamma),$$
$$\int_0^\infty \int_0^\infty \frac{dx\,dy}{1+x^4+2x^2y^2\cos 2\gamma+y^4} = \tfrac{1}{4}\pi\quad.\,F^1(\sin\gamma).$$

The first of these evaluations is given among the examples at the end of Chapter xiii. of Todhunter's "Integral Calculus," and it is this result which suggested the present paper.

3. It may also be shown that if, as before,

$$\int_0^\infty \phi\,(t^2)\,dt = A,$$

then

$$\int_0^\infty \int_0^\infty \phi\,\{(a^2x^2+b^2y^2)(\alpha^2x^2+\beta^2y^2)\}\,dx\,dy = \tfrac{1}{2}A\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{(a^2\beta^2\cos^2\theta + \alpha^2b^2\sin^2\theta)}}.$$

For
$$\frac{A}{\sqrt{(p^2\cos^2\theta+q^2\sin^2\theta)}} = \int_0^\infty \phi\,(p^2t^2\cos^2\theta + q^2t^2\sin^2\theta)\,dt,$$

and therefore

$$A\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{(p^2\cos^2\theta+q^2\sin^2\theta)}}=2\int_0^\infty\int_0^{\frac{1}{2}\pi}\phi\,(p^2r^4\cos^2\theta+q^2r^4\sin^2\theta)\,r\,dr\,dt$$

$$=2\int_0^\infty\int_0^\infty\phi\,\{(x^2+y^2)\,(p^2x^2+q^2y^2)\}\,dx\,dy.$$

Transforming the double integral by substituting αx, βy for x, y respectively, it becomes

$$=\alpha\beta\int_0^\infty\int_0^\infty\phi\,\{(\alpha^2x^2+\beta^2y^2)\,(p^2\alpha^2x^2+q^2\beta^2y^2)\}\,dx\,dy\,;$$

and therefore, putting $a=p\alpha$, $b=q\beta$, we have

$$A\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{(a^2\beta^2\cos^2\theta+\alpha^2b^2\sin^2\theta)}}=2\int_0^\infty\int_0^\infty\phi\,\{(a^2x^2+b^2y^2)(\alpha^2x^2+\beta^2y^2)\}\,dx\,dy$$

4. If
$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{(a^2\cos^2\theta+b^2\sin^2\theta)}}$$

be denoted by $F(a, b)$, so that

$$F(a,b)=\frac{1}{a}\,F^1\left(\frac{\sqrt{(a^2-b^2)}}{a}\right)=\frac{1}{b}\,F^1\left(\frac{\sqrt{(b^2-a^2)}}{b}\right),$$

then it follows from the result in the last section that

$$\tfrac{1}{2}A\,.\,F(a,b)=\int_0^\infty\int_0^\infty\phi\,\{(ax^2+by^2)\,(bx^2+ay^2)\}\,dx\,dy$$

$$=\int_0^\infty\int_0^\infty\phi\,\{(x^2+a^2y^2)\,(x^2+b^2y^2)\}\,dx\,dy.$$

5. In the process in § 1, the limits of the integration with regard to θ were 0 and π, and $\frac{1}{2}\theta$ was then replaced by θ, so that x, y were equivalent to $r\cos\frac{1}{2}\theta$, $r\sin\frac{1}{2}\theta$ of the original variables; but in § 3 the limits of integration were 0 and $\frac{1}{2}\pi$, and x, y represented $r\cos\theta$, $r\sin\theta$.

In order to connect the results derived from the two processes transform that given in § 1 by putting $\sqrt{(a\alpha)}\,.\,x$, $\sqrt{(b\beta)}\,.\,y$ for x, y; we thus have, from § 1,

$$\int_0^\infty\int_0^\infty\phi\,\{(a^2\alpha^2x^4+2a\alpha\,b\beta\,x^2y^2\cos2\gamma+b^2\beta^2y^4)\,dx\,dy=\frac{\frac{1}{2}A}{\sqrt{(a\alpha b\beta)}}\,F^1\,(\sin\gamma).$$

and from § 3,

$$\int_0^\infty\int_0^\infty\phi\,\{(a^2x^2+b^2y^2)\,(\alpha^2x^2+\beta^2y^2)\}\,dx\,dy=\tfrac{1}{2}A\,.\,F(a\beta,\,\alpha b).$$

If the expressions subject to the sign ϕ in these two formulæ are the same,
$$2a\alpha\, b\beta \cos 2\gamma = a^2\beta^2 + \alpha^2 b^2,$$
whence
$$\cos 2\gamma = \tfrac{1}{2}\left(\frac{a\beta}{\alpha b} + \frac{\alpha b}{a\beta}\right),$$
and therefore
$$\sin\gamma = i\,\frac{\frac{1}{2}(a\beta - \alpha b)}{\sqrt{(a\alpha\, b\beta)}}.$$
Thus the two formulæ are in agreement, if
$$\frac{1}{\sqrt{(a\alpha\, b\beta)}}\, F^1\left(i\,\frac{\frac{1}{2}(a\beta-\alpha b)}{\sqrt{(a\alpha b\beta)}}\right) = F(a\beta,\ \alpha b).$$
Now the left-hand member of this equation
$$= \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{\{a\alpha\, b\beta + \{\frac{1}{2}(a\beta - \alpha b)\}^2 \sin^2\theta\}}}$$
$$= \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{\{a\alpha\, b\beta \cos^2\theta + \{\frac{1}{2}(a\beta + \alpha b)\}^2 \sin^2\theta\}}}$$
$$= F\{\sqrt{(a\alpha\, b\beta)},\ \tfrac{1}{2}(a\beta + \alpha b)\},$$
and this $= F(a\beta,\ \alpha b)$ in virtue of Gauss's Theorem of the Arithmetico-geometric Mean.

6. It is thus evident that the transformations in § 1 and §3, afford a proof of this theorem of Gauss's, viz., that
$$F(a,\ b) = F\{\sqrt{(ab)},\ \tfrac{1}{2}(a+b)\}.$$
The proof may be exhibited as follows,
$$A\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{(a^2\cos^2\theta + b^2\sin^2\theta)}} = \int_0^{\infty}\int_0^{\frac{1}{2}\pi} \phi\,(a^2t^2\cos^2\theta + b^2t^2\sin^2\theta)\,dt\,d\theta,$$
$$= 2\int_0^{\infty}\int_0^{\infty} \phi\,\{(x^2+y^2)(a^2x^2+b^2y^2)\}\,dx\,dy,$$
$$= 2\int_0^{\infty}\int_0^{\infty} \phi\,\{(ax^2+by^2)(bx^2+ay^2)\}\,dx\,dy \;..(1),$$
and also
$$2A\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{(a^2\cos^2\theta + b^2\sin^2\theta)}}$$
$$= \int_0^{\infty}\int_0^{\pi} \phi\,\{a^2t^2 + 4(b^2-a^2)\,t^2\sin^2\tfrac{1}{2}\theta\cos^2\tfrac{1}{2}\theta\}\,dt\,d\theta$$
$$= 4\int_0^{\infty}\int_0^{\infty} \phi\,\{a^2(x^4+y^4) + 2(2b^2-a^2)\,x^2y^2\}\,dx\,dy \ldots (2).$$
Now, if we put $\qquad a^2 = \alpha\beta,\quad b = \frac{1}{2}(\alpha+\beta),$

the expression subject to the sign ϕ in (2)

$$= \alpha\beta\,(x^4+y^4)+(\alpha^2+\beta^2)\,x^2y^2,$$
$$= (\alpha x^2+\beta y^2)\,(\beta x^2+\alpha y^2),$$

and therefore, from (2),

$$A\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{(a^2\cos^2\theta+b^2\sin^2\theta)}} = 2\int_0^\infty\int_0^\infty \phi\,\{(\alpha x^2+\beta y^2)(\beta x^2+\alpha y^2)\}\,dx\,dy$$
$$= A\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{(\alpha^2\cos^2\theta+\beta^2\sin^2\theta)}},\ \text{from (1)},$$

that is, $$F\{\alpha\beta,\ \tfrac{1}{2}(\alpha+\beta)\} = F(\alpha,\beta).$$

7. If $$\int_0^\infty \phi\,(t^3)\,dt = B,$$

then $$\frac{B}{\sqrt[3]{c}} = \int_0^\infty \phi\,(ct^3)\,dt,$$

and therefore, putting as before $c = 1-k^2\sin^2\theta$,

$$\frac{B}{\sqrt[3]{(1-k^2\sin^2\theta)}} = 2\int_0^\infty \phi\,(r^6-k^2r^6\sin^2\theta)\,r\,dr.$$

Integrating with regard to θ between the limits 0 and π,

$$2B\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt[3]{(1-k^2\sin^2\theta)}} = 4\int_0^\infty\int_0^\infty \phi\,(r^6-4k^2r^2x^2y^2)\,dx\,dy.$$

The quantity $\sqrt{(1-k^2\sin^2\theta)}$ is usually denoted by $\Delta\theta$, so that the integral on the left-hand side of this equation

$$= \int_0^{\frac{1}{2}\pi}\frac{d\theta}{(\Delta\theta)^{\frac{2}{3}}} = \int_0^{\mathrm{K}}\frac{\operatorname{dn}u}{(\operatorname{dn}u)^{\frac{2}{3}}}\,du,$$

on putting $\theta = \operatorname{am}u$, $$= \int_0^{\mathrm{K}}\sqrt[3]{(\operatorname{dn}u)}\,du,$$

and therefore

$$\tfrac{1}{2}B\int_0^{\mathrm{K}}\sqrt[3]{(\operatorname{dn}u)}\,du = \int_0^\infty\int_0^\infty \phi\,\{x^6+y^6+(3-4k^2)\,x^2y^2\,(x^2+y^2)\}\,dx\,dy,$$
$$= \int_0^\infty\int_0^\infty \phi\left\{x^6+\frac{\sin 3\gamma}{\sin\gamma}\,x^2y^2\,(x^2+y^2)+y^6\right\}dx\,dy,$$

where, as before, $k = \sin\gamma$.

If, instead of integrating from 0 to π, we integrate from 0 to $\frac{1}{2}\pi$, or from 0 *to* $\frac{3}{2}\pi$, and substitute θ for $\frac{1}{2}\theta$, we find that, in the last equation, we may replace the expression subject to the sign ϕ by

$$x^6+(3-k^2)\,x^4y^2+(3-2k^2)\,x^2y^4+(1-k^2)\,y^6,$$

or by $$x^6+3\,(1-3k^2)\,x^4y^2+3\,(1+2k^2)\,x^2y^4+(1-k^2)\,y^6.$$

8. Similarly, if $$C = \int_0^\infty \phi(t^4)\,dt,$$

then, by integrating from 0 to π, and from 0 to 2π, and substituting θ for $\frac{1}{2}\theta$ and $\frac{1}{4}\theta$ respectively, we find that

$$\tfrac{1}{2}C\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{(\Delta\theta)}} = \tfrac{1}{2}C\int_0^{\mathrm{K}}\sqrt{(\mathrm{dn}\,u)}\,du$$

$$= \int_0^\infty\int_0^\infty \phi\,\{x^8+4\,(1-k^2)\,x^2y^2\,(x^4+y^4)+2\,(3-4k^2)\,x^4y^4+y^8\}\,dx\,dy$$

$$= \int_0^\infty\int_0^\infty \phi\,\{x^8+4\,(1-4k^2)\,x^2y^2\,(x^4+y^4)+2\,(3+16k^2)\,x^4y^4+y^8\}\,dx\,dy,$$

and so on.

9. As particular cases of the foregoing results, we have

$$\int_0^\infty\int_0^\infty \phi\,(x^4+y^4)\,dx\,dy = \tfrac{1}{2}A\,.\,F^1\left(\frac{1}{\sqrt{2}}\right),$$

$$\int_0^\infty\int_0^\infty \phi\,(x^6+y^6)\,dx\,dy = \tfrac{1}{2}B\int_0^{\mathrm{K}}\sqrt[3]{(\mathrm{dn}\,u)}\,du \quad \left(k=\frac{\sqrt{3}}{2}\right),$$

$$\int_0^\infty\int_0^\infty \phi\,(x^8-2x^4y^4+y^8)\,dx\,dy = \tfrac{1}{2}C\int_0^{\mathrm{K}}\sqrt{(\mathrm{dn}\,u)}\,du \quad (k=1),$$

$$\int_0^\infty\int_0^\infty \phi\,(x^8+x^6y^2+x^2y^6+y^8)\,dx\,dy = \tfrac{1}{2}C\int_0^{\mathrm{K}}\sqrt{(\mathrm{dn}\,u)}\,du \quad \left(k=\frac{\sqrt{3}}{2}\right),$$

$$\int_0^\infty\int_0^\infty \phi\,(x^8+14x^4y^4+y^8)\,dx\,dy = \tfrac{1}{2}C\int_0^{\mathrm{K}}\sqrt{(\mathrm{dn}\,u)}\,du \quad \left(k=\frac{1}{2}\right).$$

Since, when $k=1$,

$$\int_0^{\mathrm{K}}\sqrt{(\mathrm{dn}\,u)}\,du = \int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{(\cos\theta)}} = \sqrt{2}\,.\,F^1\left(\frac{1}{\sqrt{2}}\right),$$

the third of these equations may be written

$$\int_0^\infty\int_0^\infty \phi\,(x^8-2x^4y^4+y^8)\,dx\,dy = \frac{1}{\sqrt{2}}\,C\,.\,F^1\left(\frac{1}{\sqrt{2}}\right).$$

Also, putting $k=1$ in §7,

$$\int_0^\infty\int_0^\infty \phi\,(U)\,dx\,dy = \int_0^{\mathrm{K}}\sqrt[3]{(\mathrm{dn}\,u)}\,du \quad (k=1),$$

where $\qquad U = x^6-x^4y^2-x^2y^4+y^6,$

or $\qquad\quad = x^6+2x^4y^2+x^2y^4,$

or $\qquad\quad = x^6-6x^4y^2+9x^2y^4;$

and, when $k=1$, $\displaystyle\int_0^{\mathrm{K}}\sqrt[3]{(\mathrm{dn}\,u)}\,du = \int_0^{\frac{1}{2}\pi}\frac{d\theta}{(\cos\theta)^{\frac{1}{3}}},$

the value of which integral is*

$$\frac{3}{\sqrt[4]{3}}F^1(\sin 15^\circ) = \frac{3}{\sqrt[4]{3}}F^1\{\tfrac{1}{2}\sqrt{(2-\sqrt{3})}\}.$$

10. Taking $\phi(t) = e^{-t}$, or, as it is convenient to write it, $\exp(-t)$,

since $$\int_0^\infty \exp(-t^n)\,dt = \Gamma\left(1+\frac{1}{n}\right),$$

we have
$$A = \Gamma(1+\tfrac{1}{2}) = \tfrac{1}{2}\sqrt{\pi},$$
$$B = \Gamma(1+\tfrac{1}{3}) = \tfrac{1}{3}\Gamma(\tfrac{1}{3}),$$
$$C = \Gamma(1+\tfrac{1}{4}) = \tfrac{1}{4}\Gamma(\tfrac{1}{4});$$

and therefore

$$\int_0^\infty\int_0^\infty \exp\{-(x^4+y^4)\}\,dx\,dy = \tfrac{1}{4}\sqrt{\pi}\,.\,F^1\left(\frac{1}{\sqrt{2}}\right),$$

$$\int_0^\infty\int_0^\infty \exp\{-(x^6+y^6)\}\,dx\,dy = \tfrac{1}{6}\Gamma(\tfrac{1}{3})\,.\int_0^K \sqrt[3]{(\mathrm{dn}\,u)}\,du \qquad \left(k=\frac{\sqrt{3}}{2}\right),$$

$$\int_0^\infty\int_0^\infty \exp\{-(x^8-2x^4y^4+y^8)\}\,dx\,dy = \tfrac{1}{8}\Gamma(\tfrac{1}{4})\,.\,\sqrt{2}\,F^1\left(\frac{1}{\sqrt{2}}\right),$$

&c. &c.

Since $$\int_0^\infty\int_0^\infty \exp\{-(x^n+y^n)\}\,dx\,dy = \int_0^\infty \exp(-x^n)\,dx\,.\int_0^\infty \exp(-y^n)\,dy$$
$$= \Gamma^2\left(1+\frac{1}{n}\right),$$

the first two of these formulæ give

$$\tfrac{1}{16}\Gamma^2(\tfrac{1}{4}) = \tfrac{1}{4}\sqrt{\pi}\,.\,F^1\left(\frac{1}{\sqrt{2}}\right),$$

$$\tfrac{1}{36}\Gamma^2(\tfrac{1}{6}) = \tfrac{1}{6}\Gamma(\tfrac{1}{3})\int_0^K \sqrt[3]{(\mathrm{dn}\,u)}\,du \qquad \left(k=\frac{\sqrt{3}}{2}\right);$$

whence $$F^1\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4}\,\frac{\Gamma^2(\frac{1}{4})}{\sqrt{\pi}} = \frac{1}{4}\,\frac{\Gamma^2(\frac{1}{4})}{\Gamma(\frac{1}{2})},$$

$$\int_0^K \sqrt[3]{(\mathrm{dn}\,u)}\,du = \frac{1}{6}\,\frac{\Gamma^2(\frac{1}{6})}{\Gamma(\frac{1}{3})} \qquad \left(k=\frac{\sqrt{3}}{2}\right),$$

the first of which is a well-known result.

The integral in the second equation may be written also in the form

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt[3]{(1-\frac{3}{4}\sin^2\theta)}}, \quad = \sqrt[3]{4}\,.\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt[3]{(1+3\sin^2\theta)}}.$$

11. It may be observed that, since

$$\int_0^\infty\int_0^\infty \exp\{-(a^nx^n+b^ny^n)\}\,dx\,dy = \frac{1}{ab}\,\Gamma^2\left(1+\frac{1}{n}\right),$$

---

* Legendre, *Traité des Fonctions Elliptiques*, t. i. p. 57.

we have

$$\int_0^\infty \int_0^{\frac{1}{2}\pi} \exp\{-r^n(a^n\cos^n\theta + b^n\sin^n\theta)\}\, r\,dr\,d\theta = \frac{1}{ab}\Gamma^2\left(1+\frac{1}{n}\right);$$

and therefore
$$\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt[\frac{1}{2}n]{(a^n\cos^n\theta + b^n\sin^n\theta)}} = \frac{2}{ab}\,\frac{\Gamma^2\left(1+\frac{1}{n}\right)}{\Gamma\left(1+\frac{2}{n}\right)}$$

$$= \frac{1}{nab}\,\frac{\Gamma^2\left(\frac{1}{n}\right)}{\Gamma\left(\frac{2}{n}\right)}.$$

On the Tangents drawn from a Point to a Nodal Cubic.

By R. A. Roberts, B.A.

[*Read Jan. 13th,* 1881.]

The equation of a nodal cubic, referred to the triangle formed by the inflexional tangents, may be written $x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}=0$, or in tangential coordinates $\lambda^{-\frac{1}{2}}+\mu^{-\frac{1}{2}}+\nu^{-\frac{1}{2}}=0$. If we combine with this last equation the equation of a point $x\lambda+y\mu+z\nu=0$, we get a biquadratic which determines the tangents drawn from (x, y, z) to the cubic.

Let us consider the cubics $f = au^3+bv^3+cw^3$, $f' = a'u^3+b'v^3+c'w^3$, where $u+v+w=0$. Then the discriminant of $f+kf'$ is seen to be $(a+ka')^{-\frac{1}{2}}+(b+kb')^{-\frac{1}{2}}+(c+kc')^{-\frac{1}{2}}=0$. Let us suppose that this equation in k coincides with the biquadratic found above; we must have, then, $\lambda = a+ka'$, $\mu = b+kb'$, $\nu = c+kc'$; and, since $x\lambda+y\mu+z\nu=0$, identically,

$$\frac{x}{(bc')} = \frac{y}{(ca')} = \frac{z}{(ab')} \quad\text{.........................(1).}$$

Now, the invariants, S and T, of the equation in k are expressed in terms of the combinants, P and Q, of the cubics f and f', thus (Salmon's *Higher Algebra*, Art. 204),

$$S = 3P(P^3-24Q),$$
$$T = -(P^6-36P^3Q+216Q^2).$$

But
$$P = (bc')+(ca')+(ab'),$$
$$Q = (ab')(bc')(ca').$$

Hence, from (1),
$$P = x+y+z,$$
$$Q = xyz;$$

therefore $S = 3(x+y+z)\{(x+y+z)^3 - 24xyz\}$,

$$T = -\{(x+y+z)^6 - 36xyz\,(x+y+z)^3 + 216x^2y^2z^2\},$$

and $$\frac{S^3}{T^2} = \frac{27\lambda\,(\lambda-24)^3}{(\lambda^2-36\lambda+216)^2} \qquad \text{.......................(2)},$$

where $$(x+y+z)^3 - \lambda xyz = 0.$$

Let us calculate the invariants of the cubic $(x+y+z)^3 - \lambda xyz = 0$. We find $S' = 3\lambda^3(\lambda-24)$, $T' = -\lambda^4(\lambda^2-36\lambda+216)$. Hence we infer that the absolute invariant $\frac{S^3}{T^2}$ of the tangents drawn from any point to the nodal cubic $(x+y+z)^3 - 27xyz = 0$, is equal to the absolute invariant $\frac{S'^3}{T'^2}$ of the cubic of the system $(x+y+z)^3 - \lambda xyz = 0$, which passes through the point.

In particular, the tangents form a harmonic pencil from any point of either of the harmonic cubics $(x+y+z)^3 - 6(3 \pm \sqrt{3})\,xyz = 0$, and S vanishes for any point of the line of inflexions, and the cubic $(x+y+z)^3 - 24xyz = 0$.

2. The cubic, being referred to the triangle formed by the nodal tangents and the line of inflexions, can be written $x^3 + y^3 + 6xyz = 0$. Eliminating z between the equations of the curve and the polar conic of (x', y', z'), $U \equiv x'(x^2+2yz) + y'(y^2+2zx) + 2z'xy = 0$, we obtain

$$y'x^4 - 2x'x^3y - 6z'x^2y^2 - 2y'xy^3 + x'y^4 = 0 \qquad \text{..............(3)}.$$

Multiplying this equation (3) by $x'x^2 + y'y^2$, it becomes

$$x'y'(x^6+y^6-4x^3y^3) + (x'^2 - 6y'z')\,x^2y^4 + (y'^2 - 6z'x')\,x^4y^2$$
$$-2x'^2x^5y - 2y'^2xy^5 = 0.$$

But, from the equation of the curve,

$$x^6 + y^6 - 4x^3y^3 = 6x^2y^2(6z^2 - xy), \quad x^5y = -x^2y^2(y^2+6zx),$$
$$xy^5 = -x^2y^2(x^2+6yz);$$

hence, substituting and dividing by $3x^2y^2$, we have

$$(y'^2 - 2z'x')\,x^2 + (x'^2 - 2y'z')\,y^2 + 12x'y'z^2 + 4y'^2yz + 4x'^2zx - 2x'y'xy = 0,$$

which represents a conic passing through the points of contact of tangents from $(x', y'\ z')$ to the curve. If we call this conic V, $V + \lambda U = 0$ represents any conic passing through the points of contact of the tangents.

Hence, the locus of points, from which the tangents have their points of contact on a conic passing through two fixed points, is a cubic $U_1V_2 - V_1U_2 = 0$. By taking for the fixed points the circular points at infinity, we have the locus of the points whence the tangents have their points of contact on a circle.

3. Putting $\lambda = -2z'$ in the equation of the conic $V + \lambda U = 0$, we obtain the equation of the conic of the system which passes through (x', y', z'),

$$(y'^2 - 4z'x')\,x^2 + (x'^2 - 4y'z')\,y^2 + 12x'y'z^2 + 4\,(y'^2 - z'x')\,yz \\ + 4\,(x'^2 - y'z')\,zx - 2\,(x'y' + 2z'^2)\,xy = 0.$$

The tangent to this conic at (x', y', z') is

$$z'\,(x'^2 + 2y'z')\,x + z'\,(y'^2 + 2z'x')\,y - (x'^3 + y'^3 + 4x'y'z')\,z = 0,$$

which coincides with the tangent to the cubic

$$z'^3\,(x^3 + y^3 + 6xyz) - (x'^3 + y'^3 + 6x'y'z')\,z^3 = 0,$$

as it ought (see Salmon's *Higher Plane Curves*, Art. 169).

4. The discriminant of $V + \lambda U$ is found to be, after dividing by the Hessian, $\lambda^3 - 6z'\lambda^2 + 4\,(x'^3 + y'^3 + 6x'y'z') = 0$. By means of this result we can find the locus of the intersection of tangents at the extremities of a chord which passes through a fixed point. Forming the equation of the chords of intersection of U and V, and expressing that this equation is satisfied by the coordinates of the fixed point, we obtain the equation of the locus,

$$W^3 + 6zPW^2 - 4\,(x^3 + y^3 + 6xyz)\,P^3 = 0,$$

where

$$W = (y'^2 + 4z'x')\,x^2 + (x'^2 + 4y'z')\,y^2 - 2y'^2yz - 2x'^2zx + 2\,(6z'^2 - x'y')\,xy,$$
$$P = (x'^2 + 2y'z')\,x + (y'^2 + 2z'x')\,y + 2x'y'z,$$

(x, y, z) being now the coordinates of a point on the locus, and (x', y', z') of the fixed point.

When the fixed point is on the curve, W becomes divisible by P, and therefore also the locus, which becomes, in this case,

$$4x'^3y'^3\,(x^3 + y^3) + (x'^2x + y'^2y)^3 - 6x'y'\,(x'^2x - y'^2y)^2\,z = 0.$$

Thursday, March 10th, 1881.

S. ROBERTS, Esq., F.R.S., President, in the Chair.

Prof. Cayley read a paper "On the Flexure and Equilibrium of a Skew Surface."

Mr. Tucker communicated portions of papers, viz., "An Application of Elliptic Functions to the Nodal Cubic," Mr. R. A. Roberts; and "Note on Prof. Peirce's Probability Notation of 1867," Mr. Hugh McColl.

Mr. Glaisher, Vice-President, having taken the Chair, the President communicated a Theorem which was the Analogue in Space of the Theorem relating to three Circles intersecting in a Point.

The following presents were made to the Library :—

"Educational Times," March, 1881.

"An Introduction to the Ancient and Modern Geometry of Conics," by C. Taylor, M.A. : from the Author.

"American Journal of Mathematics," Vol. iii., No. 3.

"Various Papers and Notes that have appeared in the 'Quarterly Journal of Mathematics,' and the 'Messenger of Mathematics,' during the year 1880, by J. W. L. Glaisher, F.R.S."

"On the Method of Least Squares," from the "Monthly Notices of the Royal Astronomical Society," Vol. xl., No. 9, and Vol. xvi., No. 1.

"Note on a Method of obtaining the q-formula for the Sine-amplitude in Elliptic Functions." (Proceedings of London Mathematical Society, Vol. xi., No. 158.)

"On the Deduction of Trigonometrical from Elliptic Function Formulæ," and two other Notes. (Report of British Association, 1880.)

Papers extracted from the "Proceedings of the Cambridge Philosophical Society," Vol. iii., Pt. viii.

"Report of Committee of British Association on Mathematical Tables." (Report of British Association, 1880.)

The above from the Author, J. W. L. Glaisher, F.R.S.

"Bulletins de l'Académie Royale .... de Belgique," 49me année, 2e série, tomes xlvi., xlvii., xlviii., xlix., l., 1880.

"Annuaire de l'Académie Royale .... de Belgique," 1879, 1880, 1881; Bruxelles.

A Note on Prof. C. S. Peirce's Probability Notation of 1867. *By* Hugh McColl, B.A.

[The Secretaries were directed by the Council to state that they had received a note from Mr. McColl which showed them that the apparent coincidence of notation, in some few particulars, between himself and Prof. Peirce, was entirely accidental, and that Mr. McColl was not at the time acquainted with Prof. Peirce's paper. In fact, the revised fourth paper ("On Probability Notation," Proceedings of London Mathematical Society, Vol. xi., No. 163) was communicated to the Society about nine months before the Author read Prof. Peirce's paper.]

On the Flexure and Equilibrium of a Skew Surface.

By Prof. CAYLEY.

[*Read March* 10*th*, 1881.]

The skew surface is taken to be such that the strip between two consecutive generating lines is rigid, and that the flexure takes place by the rotation of the strips about the generating lines successively. The theory of the flexure is well known, but I am not aware that the theory of the equilibrium of such a surface, when acted upon by any given forces, has been considered; it is, however, a question which presents itself naturally in connexion with those relating to other continuous bodies treated of in the *Mécanique Analytique*, and forms a good example of the principles made use of.

To begin with the mechanical theory: we may regard the forces as acting on the generating lines regarded as material lines; and if for an element of mass dm, coordinates (x, y, z) of a particular generating line G, the forces parallel to the axes are X', Y', Z', then the corresponding term in the equation of equilibrium is

$$S\,(X'\delta x + Y'\delta y + Z'\delta z)\,dm\,;$$

and observing that there are (as will afterwards appear) *five* geometrical conditions, which I represent by $U_1 = 0$, $U_2 = 0$, ... $U_5 = 0$, the equation of equilibrium is

$$S\,\{(X'\delta x + Y'\delta y + Z'\delta z)\,dm + T_1\delta U_1 + T_2\delta U_2 + T_3\delta U_3 + T_4\delta U_4 + T_5\delta U_5\} = 0,$$

where T_1, T_2, ... T_5 are the indeterminate multipliers, representing colligation-forces which correspond to the five geometrical conditions respectively.

Taking (ξ, η, ζ) for the coordinates of a particular point P on the generating line; p, q, r for the cos-inclinations of the line (whence $U_1 = p^2+q^2+r^2-1 = 0$ is one of the geometrical relations), and ρ for the distance of dm from P, we have

$$x,\ y,\ z = \xi+\rho p,\quad \eta+\rho q,\quad \zeta+\rho r,$$

$$\delta x,\ \delta y,\ \delta z = \delta\xi+\rho\delta p,\ \delta\eta+\rho\delta q,\ \delta\zeta+\rho\delta r.$$

The summation S extends first to the different points of the generating line, and then to the different generating lines; applying it first to the particular generating line, we write

$$\begin{array}{cccccc} SX'dm, & SY'dm, & SZ'dm, & SX'\rho dm, & SY'\rho dm, & SZ'\rho dm \\ = X, & Y, & Z, & L, & M, & N, \end{array}$$

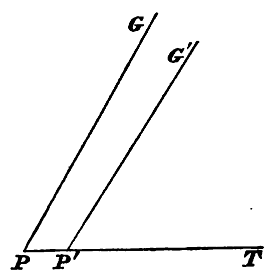
where X, Y, Z are the whole forces, and L, M, N the whole moments about the point P, for the generating line G; retaining the same

summatory symbol S, as now referring to the different generating lines, the equation becomes

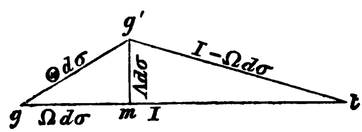
$$S\{X\delta\xi+Y\delta\eta+Z\delta\zeta+L\delta p+M\delta q+N\delta r+T_1\delta U_1+\ldots+T_5\delta U_5\}=0.$$

We have now to consider the geometrical theory of the flexure. Taking on the skew surface an arbitrary curve cutting each generating G in a point P, coordinates (ξ,η,ζ), and taking σ for the distance along the curve of the point P from a fixed point of the curve; also p, q, r, as before, for the cos-inclinations of the generating line G, then when the surface is in a determinate state, $\xi, \eta, \zeta, p, q, r$ are given functions of σ; but these functions vary with the flexure of the surface, with, however, certain relations unaffected by the flexure; and the problem is to find first these relations. As already mentioned, one of them is $p^2+q^2+r^2-1=0$.

Taking P' as the consecutive point on the curve, so that the direction of the element PP' is that of the tangent PT at P, it is convenient to write l, m, n for the cosine-inclinations of the tangent; we have, it is clear,
$$l,\ m,\ n=\frac{d\xi}{d\sigma},\ \frac{d\eta}{d\sigma},\ \frac{d\zeta}{d\sigma};\quad l^2+m^2+n^2-1=0.$$
The conditions in order to the rigidity of the strip, are that the angles GPP', $G'P'P$ $(=180^\circ-G'P'T)$, and the inclination $G'P'$ to GP, shall have given values; variable it may be from strip to strip—that is, these values must be given functions of σ. Taking $\angle\, GPT=I$, the value of $G'P'T$ can differ only infinitesimally from that of GPT, and we take it to be $G'P'T=I-\Omega\,d\sigma$; also the inclination GP to $G'P'$ is an infinitesimal, $=\Theta\,d\sigma$: we have I, Ω, Θ given functions of σ. It is to be remarked that these conditions imply, inclination of $G'P'$ to tangent plane GPT at P has a given value $\Lambda\,d\sigma$; in fact, if through P we draw a line $P\gamma$ parallel to $P'G'$, then, if P is regarded as the centre of a sphere which meets PG, $P\gamma$, PT in the points g, g', t respectively, we have a



spherical triangle $gg't$, the sides of which are $I-\Omega\,d\sigma$, I, and $\Theta\,d\sigma$, and of which the perpendicular gm is $=\Lambda\,d\sigma$; we have thus an infinitesimal right-angled triangle, the base and altitude of which are



$\Omega\,d\sigma$, $\Lambda\,d\sigma$, and the hypothenuse is $\Theta\,d\sigma$; whence $\Theta^2=\Omega^2+\Lambda^2$. In the case of the developable surface $\Lambda=0$ and $\Theta=\Omega$. It may be remarked that, when the curve on the skew surface is the line of striction,

we have $\Omega=0$; in fact, taking P to be on the line of striction, the line

$$\frac{X-\xi}{qr'-q'r}=\frac{Y-\eta}{rp'-r'p}=\frac{Z-\zeta}{pq'-p'q},$$

through (ξ, η, ζ) at right angles to the two generating lines, meets the consecutive generating line $X, Y, Z = \xi'+\rho p', \eta'+\rho q', \zeta'+\rho r'$; and the condition that this may be so is easily found to be $\Omega = 0$.

Take, for a moment, p', q', r' for the cos-inclinations of the consecutive generating line $P'G'$; we have

$$\begin{aligned} lp+mq+nr &= \cos I, \\ lp'+mq'+nr' &= \cos(I-\Omega d\sigma), \\ pp'+qq'+rr' &= \cos\Theta\, d\tau; \end{aligned}$$

and then writing $p', q', r'=p+dp, q+dq, r+dr$, and observing that the equation $p'^2+q'^2+r'^2=1$ gives

$$p\,dp+q\,dq+r\,dr=-\tfrac{1}{2}(dp^2+dq^2+dr^2),$$

these equations and the before-mentioned two equations become

$$\begin{array}{ll} (U_1) & p^2+q^2+r^2-1=0, \\ (U_2) & l^2+m^2+n^2-1=0, \\ (U_3) & lp+mq+nr-\cos I=0, \\ (U_4) & l\,dp+m\,dq+n\,dr-\Omega\sin I\,d\sigma=0, \\ (U_5) & dp^2+dq^2+dr^2-\Theta^2 d\sigma^2=0, \end{array}$$

which equations, considering therein l, m, n as standing for their values $\frac{d\xi}{d\sigma}, \frac{d\eta}{d\sigma}, \frac{d\zeta}{d\sigma}$, are the geometrical relations which connect the six variables $\xi, \eta, \zeta, p, q, r$, considered as functions of σ. And in these equations I, Ω, Θ denote given functions of σ, invariable by any flexure of the surface.

To complete the geometrical theory, it is to be observed that we can by flexure bring the generating lines of the surface to be parallel to those of any given cone $C(p, q, r)=0$, where $C(p, q, r)$ denotes a homogeneous function of (p, q, r). Hence, joining to the foregoing five equations this new equation

$$C(p, q, r)=0,$$

these six equations determine $\xi, \eta, \zeta, p, q, r$ as functions of σ. To make the solution completely determinate, we have only to assume for the point P, which corresponds, say, to the value $\sigma=0$, a position in space at pleasure, and to take the corresponding generating line PG parallel to a generating line, at pleasure, of the cone.

As an example, writing γ to denote an arbitrary constant angle, if

the invariable conditions are

$$I=\gamma,\quad \Theta=\sin\gamma,\quad \Omega=0;$$

then the five equations are

$$\begin{aligned}
p^2+\ q^2\ +r^2-\quad 1\quad &=0,\\
l^2+\ m^2+n^2-\quad 1\quad &=0,\\
lp+\ mq\ +nr-\ \cos\gamma\ &=0,\\
dp^2+dq^2+dr^2-\sin^2\gamma\, d\sigma^2 &=0,\\
l\,dp+m\,dq+n\,dr\qquad &=0;
\end{aligned}$$

and if we assume *first* $C(p,q,r)=p^2+q^2-r^2\tan^2\gamma,=0$; and *secondly* $C(p,q,r)=r,=0,$—

Then, in the former case, we find the solution

$$p,q,r=-\sin\gamma\sin\sigma,\ \sin\gamma\cos\sigma,\ \cos\gamma;$$

$$\xi,\eta,\zeta=\quad\cos\sigma,\sin\sigma,0;$$

giving $\quad x,y,z=\cos\sigma-\rho\sin\gamma\sin\sigma,\ \sin\sigma+\rho\sin\gamma\cos\sigma,\ \cos\gamma;$

and consequently $\qquad x^2+y^2-z^2\tan^2\gamma=0,$

the hyperboloid of revolution. And, in the latter case,

$$p,q,r=\cos(\sigma\sin\gamma),\ \sin(\sigma\sin\gamma),\ 0,$$

$$\xi,\eta,\zeta=\cot\gamma\sin(\sigma\sin\gamma),\ -\cot\gamma\cos(\sigma\sin\gamma),\ \sigma\sin\gamma,$$

that is, $\qquad x,y=\cot\gamma\sin z+\rho\cos z,\ -\cot\gamma\cos z+\rho\sin z,$

whence $\qquad x\sin z-y\cos z=\cot\gamma,$

a skew helicoid generated by horizontal tangents of the cylinder $x^2+y^2=\cot^2\gamma$. This is a known deformation of the hyperboloid.

Returning now to the mechanical problem, we have to consider the terms

$$\begin{aligned}
S\ .\ &T_1\delta\,.\,\tfrac{1}{2}(p^2+q^2+r^2-1)\\
&+T_2\delta\,.\,\tfrac{1}{2}(l^2+m^2+n^2-1)\\
&+T_3\delta\,(lp+mq+nr-\cos I)\\
&+T_4\delta\left(l\frac{dp}{d\sigma}+m\frac{dq}{d\sigma}+n\frac{dr}{d\sigma}-\Omega\sin I\right)\\
&+T_5\delta\,.\,\tfrac{1}{2}\left\{\left(\frac{dp}{d\sigma}\right)^2+\left(\frac{dq}{d\sigma}\right)^2+\left(\frac{dr}{d\sigma}\right)^2-\Theta^2\right\}.
\end{aligned}$$

The first term gives, under the sign S,

$$T_1(p\,\delta p+T_1q\,\delta q+T_1r\,\delta r).\qquad (*)$$

The second term gives, in the first instance,

$$\frac{T_2}{d\sigma}(l\,d\,\delta\xi+m\,d\,\delta\eta+n\,d\,\delta\zeta);$$

or, since in general

$$S\Omega d\delta\xi = \Omega''\delta\xi'' - \Omega'\delta\xi' + S(-d\Omega\,.\,\delta\xi),$$

then, attending only to the terms under the sign S, these are

$$= -\frac{d}{d\sigma}T_2 l\,.\,\delta\xi - \frac{d}{d\sigma}T_2 m\,.\,\delta\eta - \frac{d}{d\sigma}T_2 n\,.\,\delta\zeta. \qquad (*)$$

The third term gives $\quad T_3(l\,\delta p + m\,\delta q + n\,\delta r) \qquad (*)$

$$+T_3(p\,\delta l + q\,\delta m + r\,\delta n),$$

where the second line,

$$= \frac{T_3}{d\sigma}(p\,d\delta\xi + q\,d\delta\eta + r\,d\delta\zeta),$$

attending only to the terms under the sign S, gives

$$-\frac{d}{d\sigma}T_3 p\,.\,\delta\xi - \frac{d}{d\sigma}T_3 q\,.\,\delta\eta - \frac{d}{d\sigma}T_3 r\,.\,\delta\zeta. \qquad (*)$$

The fourth line gives

$$T_4\left(\frac{dp}{d\sigma}\delta l + \frac{dq}{d\sigma}\delta m + \frac{dr}{d\sigma}\delta n\right)$$

$$+T_4\left(\frac{l}{d\sigma}d\delta p + \frac{m}{d\sigma}d\delta q + \frac{n}{d\sigma}d\delta r\right),$$

where the first line, written under the form

$$\frac{T_4}{d\sigma}\left(\frac{dp}{d\sigma}d\,\delta\xi + \frac{dq}{d\sigma}d\,\delta\eta + \frac{dr}{d\sigma}d\,\delta\zeta\right),$$

and attending only to the terms under the sign S, gives

$$-\frac{d}{d\sigma}\left(T_4\frac{dp}{d\sigma}\right).\,\delta\xi - \frac{d}{d\sigma}\left(T_4\frac{dq}{d\sigma}\right).\,\delta\eta - \frac{d}{d\sigma}\left(T_4\frac{dr}{d\sigma}\right).\,\delta\zeta, \qquad (*)$$

and the second line, attending in like manner only to the terms under the sign S, gives

$$-\frac{d}{d\sigma}T_4 l\,.\,\delta p - \frac{d}{d\sigma}T_4 m\,.\,\delta q - \frac{d}{d\sigma}T_4 n\,.\,\delta r. \qquad (*)$$

The fifth line, written under the form

$$\frac{T_5}{d\sigma}\left(\frac{dp}{d\sigma}d\,\delta p + \frac{dq}{d\sigma}d\,\delta q + \frac{dr}{d\sigma}d\,\delta r\right), \qquad (*)$$

and attending only to the terms under the sign S, gives

$$-\frac{d}{d\sigma}T_5\frac{dp}{d\sigma}.\,\delta p - \frac{d}{d\sigma}T_5\frac{dq}{d\sigma}.\,\delta q - \frac{d}{d\sigma}T_5\frac{dq}{d\sigma}.\,\delta r; \qquad (*)$$

where in each case I have marked with an asterisk the lines which present themselves in the final result.

Hence, joining to the foregoing the force-terms

$$X\delta\xi + Y\delta\eta + Z\delta\zeta + L\delta p + M\delta q + N\delta r, \qquad (*)$$

and equating to zero the coefficients of $\delta\xi$, $\delta\eta$, $\delta\zeta$, δp, δq, δr respectively, we have

$$\left\{\begin{array}{llllll}
0 = X & . & -\frac{d}{d\sigma}T_2 l & -\frac{d}{d\sigma}T_3 p & -\frac{d}{d\sigma}T_4\frac{dp}{d\sigma}, & \\
0 = Y & . & -\frac{d}{d\sigma}T_2 m & -\frac{d}{d\sigma}T_3 q & -\frac{d}{d\sigma}T_4\frac{dq}{d\sigma}, & \\
0 = Z & . & -\frac{d}{d\sigma}T_2 n & -\frac{d}{d\sigma}T_3 r & -\frac{d}{d\sigma}T_4\frac{dr}{d\sigma}, & \\
0 = L + T_1 p & . & + & T_3 l & -\frac{d}{d\sigma}T_4 l & -\frac{d}{d\sigma}T_5\frac{dp}{d\sigma}, \\
0 = M + T_1 q & . & + & T_3 m & -\frac{d}{d\sigma}T_4 m & -\frac{d}{d\sigma}T_5\frac{dq}{d\sigma}, \\
0 = N + T_1 r & . & + & T_3 n & -\frac{d}{d\sigma}T_4 n & -\frac{d}{d\sigma}T_5\frac{dr}{d\sigma},
\end{array}\right.$$

where it will be recollected that l, m, n stand for $\frac{d\xi}{d\sigma}, \frac{d\eta}{d\sigma}, \frac{d\zeta}{d\sigma}$, the variables being ξ, η, ζ, p, q, r, and σ. The elimination of $T_1, T_2, \ldots T_5$ from the six equations should lead to a relation between ξ, η, ζ, p, q, r, which, with the foregoing five relations, would determine the six variables ξ, η, ζ, p, q, r in terms of σ.

In particular, the forces and moments X, Y, Z, L, M, N may all of them vanish; assuming that $T_1, T_2, \ldots T_5$ do not all of them vanish, we still have the sixth relation, which (with the foregoing five relations) determines ξ, η, ζ, p, q, r in terms of σ; and it is to be remarked that the problem in question of the figure of equilibrium of the skew surface not acted upon by any forces, is analogous to that of the geodesic line in space; only whilst here the solution is, curve a straight line, the solution for the case of the skew surface depends upon equations of a complex enough form; in the case of the developable surface the required figure is of course the plane.

Thursday, April 14th, 1881.

S. ROBERTS, Esq., F.R.S., President, in the Chair.

The Chairman briefly alluded, in feeling terms, to the loss the Society had sustained by Mr. T. Cotterill's death.

The following communications were made:—

"On the Geodesic Curvature of a Curve on a Surface," Prof. Cayley.

"On Operative Symbols in the Differential Calculus," Prof. M. W. Crofton.

"Note on the Resolution in Factors of Numbers differing but slightly from π," Mr. J. W. L. Glaisher.

"On the nature of the Quadric represented by the general Equation of the Second Degree in Tetrahedral Co-ordinates," and "On the Five Focal Quadrics of a Cyclide," Mr. H. Hart.

"The Discrimination of the Maximum or Minimum Path of a Ray of Light reflected at a given Curve," Mr. H. M. Taylor.

"On certain Tetrahedra specially related to Four Spheres meeting in a Point," and "Historical Note on Dr. Graves's Note on Confocal Conics," the President.

The following presents were received:—

"Educational Times," April, 1881.

"Catalogue of Books in the General Library, and in the South Library at University College, London, with an Appendix," three Vols.; London, 1879.

"Bulletin des Sciences Mathématiques et Astronomiques," Tome iv., Septembre, 1880.

"Crelle's Journal," 90th Band, 3es and 4es Hefte; Berlin, 1881.

"Journal of Institute of Actuaries," Vol. xxii., Pt. v., No. cxxi., Oct. 1880; with List of Members and Catalogue of the Library of the Institute of Actuaries, Nov. 1880.

"Monatsbericht," Nov. 1880; Berlin, 1881.

"Exposition Géométrique des Propriétés générales des Courbes," par C. Ruchonnet (de Lausanne), 4^{me} Edition; Paris, 1880.

"Éléments de Calcul Approximatif," par C. Ruchonnet (de Lausanne), 3^{me} Edition; Paris, 1880.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome xv., 3^{me} Livraison, 4^{me} Livraison, and 5^{me} Livraison; Harlem, 1880.

"Tidsskrift for Mathematik," udgivet af H. G. Zeuthen; Fjerde Rœkke, fjerde Aargang, første, andet, tredie, fjerde, femte, sjette Hefte; Kjøbenhavn, 1880.

"Beiblätter zu den Annalen der Physik und Chemie," No. 3, 1881, Band v., Stück 3.

"Atti della R. Accademia dei Lincei—Transunti," Vol. v., Fasc. 8°; Roma, 1881.

"Proceedings of the Royal Society," Vol. xxxi., No. 210.

"Journal de l'École Polytechnique," 48^{me} Cahier, Tome xxix.; Paris, 1880.

"Ueber gewisse Theilwerthe des Θ-Function," von F. Klein ("Math. Annalen," xvii. Band, pp. 565—574).

"United States Naval Observatory—Total Eclipse, July 29th, 1878, and Jan. 11th, 1880;" Washington, 1880.

"Greek Geometry from Thales to Euclid," Pt. ii., by G. J. Allman, LL.D. (from "Hermathena," Vol. iv., No. vii.): from the Author.

On the Geodesic Curvature of a Curve on a Surface. *By* Prof. Cayley.

[*Read April* 14*th*, 1881.]

There is contained in Liouville's Note II. to his edition of Monge's "Application de l'Analyse à la Géométrie" (Paris, 1850), see pp. 574 and 575, the following formula,

$$\frac{1}{\rho} = -\frac{di}{ds} + \frac{1}{2G\sqrt{E}}\frac{dG}{du}\cos i - \frac{1}{2E\sqrt{G}}\frac{dE}{dv}\sin i,$$
$$= -\frac{di}{ds} + \frac{\cos i}{\rho_2} + \frac{\sin i}{\rho_1},$$

which gives the radius of geodesic curvature of a curve upon a surface when the position of a point on the surface is defined by the parameters u, v, belonging to a system of orthotomic curves; or, what is the same thing, such that $ds^2 = Edu^2 + Gdv^2$. Writing with Gauss p, q instead of u, v, I propose to obtain the corresponding formula in the general case where the parameters p, q are such that

$$ds^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2.$$

I call to mind that, if PQ, PQ' are equal infinitesimal arcs on the given curve and on its tangent geodesic, then the radius of geodesic curvature ρ is, by definition, a length ρ such that $2\rho\,.\,QQ' = \overline{PQ}^2$. More generally, if the curves on the surface are any two curves which touch each other, then ρ as thus determined is the radius of relative curvature of the two curves.

The notation is that of the Memoir, "Disquisitiones generales circa superficies curvas" (1827), Gauss, Werke, t. iii.; see also my paper "On Geodesic Lines, in particular those of a Quadric Surface," Proc. Lond. Math. Society, t. iv. (1872), pp. 191—211; and Salmon's "Solid Geometry," 3rd ed., 1874, pp. 251 *et seq.* The coordinates (x, y, z) of a point on the surface are taken to be functions of two independent parameters p, q; and we then write

$$dx + \tfrac{1}{2}d^2x = a\,dp + a'\,dq + \tfrac{1}{2}(\alpha\,dp^2 + 2\alpha'\,dp\,dq + \alpha''\,dq^2),$$
$$dy + \tfrac{1}{2}d^2y = b\,dp + b'\,dq + \tfrac{1}{2}(\beta\,dp^2 + 2\beta'\,dp\,dq + \beta''\,dq^2),$$
$$dz + \tfrac{1}{2}d^2z = c\,dp + c'\,dq + \tfrac{1}{2}(\gamma\,dp^2 + 2\gamma'\,dp\,dq + \gamma''\,dq^2).$$

$$E,\ F,\ G = a^2 + b^2 + c^2,\ aa' + bb' + cc',\ a'^2 + b'^2 + c'^2;\ V = EG - F^2;$$

and therefore $\qquad ds^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2,$

where E, F, G are regarded as given functions of p and q.

To determine a curve on the surface, we establish a relation between the two parameters p, q, or, what is the same thing, take p, q to be

functions of a single parameter θ; and we write as usual p', p'', q', etc., to denote the differential coefficients of p, q, etc., in regard to θ; we write also E_1, E_2, etc., to denote the differential coefficients $\frac{dE}{dp}$, $\frac{dE}{dq}$, etc. In the first instance, θ is taken to be an arbitrary parameter, but we afterwards take it to be the length s of the curve from a fixed point thereof.

First formula for the radius of relative curvature.

Consider any two curves touching at the point P, coordinates (x, y, z) which are regarded as given functions of (p, q); where (p, q) are for the one curve given functions, and for the other curve other given functions of θ.

The coordinates of a consecutive point for the one curve are then $x+dx+\frac{1}{2}d^2x$, $y+dy+\frac{1}{2}d^2y$, $z+dz+\frac{1}{2}d^2z$, where $dp = p'd\theta+\frac{1}{2}p''d\theta^2$, $dq = q'd\theta+\frac{1}{2}q''d\theta^2$; hence these coordinates are

$$x+(ap'+a'q')\,d\theta+\tfrac{1}{2}\,(ap'^2+2a'p'q'+a''q'^2)\,d\theta^2+\tfrac{1}{2}\,(ap''+a'q'')\,d\theta^2,$$
$$\vdots$$

and for the other curve they are in like manner

$$x+(ap'+a'q')\,d\theta+\tfrac{1}{2}\,(ap'^2+2a'p'q'+a''q'^2)\,d\theta^2+\tfrac{1}{2}\,(aP''+a'Q'')\,d\theta^2,$$
$$\vdots$$

the only difference being in the terms which contain the second differential coefficients, p'', q'' for the first curve, and P'', Q'' for the second curve. Hence the differences of the coordinates are

$$\tfrac{1}{2}\,\{a\,(p''-P'')+a'\,(q''-Q'')\}\,d\theta^2,\quad \tfrac{1}{2}\,\{b\,(p''-P'')+b'\,(q''-Q'')\}\,d\theta^2,$$
$$\tfrac{1}{2}\,\{c\,(p''-P'')+c'\,(q''-Q'')\}\,d\theta^2,$$

and consequently the distance QQ' of the two consecutive points Q, Q' is

$$= \tfrac{1}{2}\sqrt{(E,\ F,\ G\between p''-P'',\ q''-Q'')^2}\,d\theta^2,$$

the squared arc $\overline{PQ}^2$ is

$$= (E,\ F,\ G\between p',\ q')^2\,d\theta^2;$$

and hence, if as before $2\rho\,.\,QQ' = \overline{PQ}^2$, that is, $\frac{1}{\rho} = 2QQ' \div \overline{PQ}^2$, then

$$\frac{1}{\rho} = \frac{\sqrt{(E,\ F,\ G\between p''-P'',\ q''-Q'')^2}}{(E,\ F,\ G\between p',\ q')^2},$$

the required formula for ρ.

Second formula for the radius of relative curvature.

We now take the variable θ to be the length s of the curve measured from a fixed point thereof, so that p', p'', etc. denote $\frac{dp}{ds}$, $\frac{d^2p}{ds^2}$, etc. We

have therefore $1 = (E, F, G \between p', q')^2$,

and the formula becomes

$$\frac{1}{\rho} = \sqrt{(E, F, G \between p''-P'', q''-Q'')^2}.$$

But, differentiating the above equation as regards the curve, we find

$$0 = 2\,(E, F, G \between p', q' \between p'', q'') + (\dot{E}, \dot{F}, \dot{G} \between p', q')^2,$$

where $\dot{E}$, $\dot{F}$, $\dot{G}$ are used to denote the complete differential coefficients $E_1p'+E_2q'$, etc. And similarly, differentiating in regard to the tangent geodesic, we obtain

$$0 = 2\,(E, F, G \between p', q' \between P'', Q'') + (\dot{E}, \dot{F}, \dot{G} \between p', q')^2;$$

and hence, taking the difference of the two equations,

$$0 = (E, F, G \between p', q' \between p''-P'', q''-Q'').$$

Hence, in the equation for $\frac{1}{\rho}$, the function under the radical sign may be written $(E, F, G \between p', q')^2 . (E, F, G \between p''-P'', q''-Q'')^2$

$$-\{(E, F, G \between p', q' \between p''-P'', q''-Q'')\}^2,$$

which is identically

$$= (EG-F^2)\,\{p'(q''-Q'')-q'(p''-P'')\}^2.$$

Hence, extracting the square root, and for $\sqrt{EG-F^2}$ writing V, we have

$$\frac{1}{\rho} = V\,\{p'(q''-Q'')-q'(p''-P'')\},$$

or say

$$\frac{1}{\rho} = V\,(p'q''-q'p'')-V\,(p'Q''-q'P''),$$

which is the new formula for the radius of relative curvature.

Formula for the radius of geodesic curvature.

In the paper "On Geodesic Lines, &c.," p. 195, writing $EG-F^2=V^2$, and P'', Q'' in place of p'', q'', the differential equation of the geodesic line is obtained in the form

$$\begin{aligned} &(Ep'+Fq')\,\{(2F_1-E_2)\,p'^2+2G_1p'q'+G_2q'^2\} \\ &-(Fp'+Gq')\,\{E_1p'^2+2E_2p'q'+(2F_2-G_1)\,q'^2\} \\ &+2V^2\,(p'Q''-q'P'') = 0; \end{aligned}$$

or, denoting by Ω the first two lines of this equation, we have

$$V\,(p'Q''-q'P'') = -\frac{\frac{1}{2}}{V}\,\Omega.$$

The foregoing equation gives therefore, for the radius of geodesic curvature,
$$\frac{1}{\rho} = V(p'q''-p''q') + \frac{\frac{1}{2}}{V}\,\Omega,$$
which is an expression depending only upon p', q', the first differential coefficients (common to the curve and geodesic), and on p'', q'', the second differential coefficients belonging to the curve.

Observe that Ω is a cubic function of p', q': we have
$$\Omega = (\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{D}\between p',\ q')^3,$$
the values of the coefficients being
$$\begin{aligned}
\mathfrak{A} &= 2EF_1 - EE_2 - FE_1,\\
\mathfrak{B} &= 2EG_1 + 2FF_1 - 3FE_2 - GE_1,\\
\mathfrak{C} &= EG_2 + 3FG_1 - 2FF_2 - 2GE_2,\\
\mathfrak{D} &= FG_2 - 2GF_2 + GG_1.
\end{aligned}$$

The Special Curves, p = constant and q = constant.

Consider the curve p = const.; for this curve $p'=0$, $p''=0$; therefore also $Gq'^2=1$; and, if R be the radius of geodesic curvature, then
$$\frac{1}{R} = \frac{\frac{1}{2}}{V}\,\mathfrak{D}q'^3,\ = \frac{\frac{1}{2}}{V}\,\frac{\mathfrak{D}}{G\sqrt{G}}.$$
Similarly for the curve q = const. Here $q'=0$, $q''=0$; therefore $Ep'^2=1$, and, if S be the radius of geodesic curvature, then
$$\frac{1}{S} = \frac{\frac{1}{2}}{V}\,\mathfrak{A}p'^3,\ = \frac{\frac{1}{2}}{V}\,\frac{\mathfrak{A}}{E\sqrt{E}}.$$
These values of R and S are interesting for their own sakes, and they will be introduced into the expression for the radius of geodesic curvature ρ of the general curve.

Transformed Formula for the Radius of Geodesic Curvature.

From the values of $\frac{1}{R}$, $\frac{1}{S}$, we have
$$\frac{1}{\rho} - \frac{q'\sqrt{G}}{R} - \frac{p'\sqrt{E}}{S} = V(p'q''-p''q') + \frac{\frac{1}{2}}{V}\left\{\Omega - \frac{\mathfrak{A}}{E}\,p' - \frac{\mathfrak{D}}{G}\,q'\right\},$$
where the term in { } is
$$= \mathfrak{A}p'^3 - \frac{\mathfrak{A}}{E}\,p' + \mathfrak{B}p'^2q' + \mathfrak{C}p'q'^2 + \mathfrak{D}q'^3 - \frac{\mathfrak{D}}{G}\,q'.$$

The terms in $\mathfrak{A}$ are
$$= -\frac{\mathfrak{A}}{E}\,p'(1-Ep'^2), \quad = -\frac{\mathfrak{A}}{E}\,p'(2Fp'q'+Gq'^2),$$

and those in $\mathfrak{D}$ are

$$=-\frac{\mathfrak{D}}{G}q'(1-Gq'^2), \quad =-\frac{\mathfrak{D}}{G}q'(Ep'^2+2Fp'q').$$

Hence the whole expression contains the factor $p'q'$, and is, in fact,

$$=p'q'\left\{p'\left(\mathfrak{B}-\frac{2\mathfrak{A}F}{E}-\frac{\mathfrak{D}E}{G}\right)+q'\left(\mathfrak{B}-\frac{\mathfrak{A}G}{E}-\frac{2\mathfrak{D}F}{G}\right)\right\};$$

or substituting for $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ their values, this is

$$=p'q'\left\{p'\left(-GE_1+EG_1+\frac{2F^2E_1}{E}-2FF_1-FE_2+2EF_2-\frac{EFG_2}{G}\right)\right.$$
$$\left.+q'\left(-GE_2+EG_2-\frac{2F^2G_2}{G}+2FF_2+FG_1-2GF_1+\frac{GFE_1}{E}\right)\right\},$$

say this is $\qquad =p'q'(Lp'+Mq');$

and the formula thus is

$$\frac{1}{\rho}-\frac{q'\sqrt{G}}{R}-\frac{p'\sqrt{E}}{S}=V(p'q''-p''q')+\frac{\frac{1}{2}}{V}p'q'(Lp'+Mq').$$

Taking ϕ, θ to be the inclination of the curve to the curves $q=$const., $p=$ const., respectively, and $\omega\,(=\phi+\theta)$ the inclination of these two curves to each other, then

$$\cos\phi=\frac{Fp'+Gq'}{\sqrt{G}},\quad \cos\theta=\frac{Ep'+Fq'}{\sqrt{E}},\quad \cos\omega=\frac{F}{\sqrt{EG}},$$
$$\sin\phi=\frac{Vp'}{\sqrt{G}},\quad \sin\theta=\frac{Vq'}{\sqrt{E}},\quad \sin\omega=\frac{V}{\sqrt{EG}};$$

hence $\dfrac{\sin\phi}{\sin\omega}=p'\sqrt{E}$, $\dfrac{\sin\theta}{\sin\omega}=q'\sqrt{G}$, and the formula may also be written

$$\frac{1}{\rho}-\frac{\sin\theta}{\sin\omega}\frac{1}{R}-\frac{\sin\phi}{\sin\omega}\frac{1}{S}=V(p'q''-p''q')+\frac{\frac{1}{2}}{V}p'q'(Lp'+Mq').$$

The Orthotomic Case $F=0$, *or* $ds^2=Edp^2+Gdq^2$.

The formula becomes in this case much more simple. We have $1=Ep'^2+Gq'^2$, $V=\sqrt{EG}$, $\omega=90°$, $\sin\theta=\cos\phi$; and the term $Lp'+Mq'$ becomes $=E\dot{G}-\dot{E}G$, if, as before, $\dot{E}$, $\dot{G}$ denote the complete differential coefficients $E_1p'+E_2q'$ and $G_1p'+G_2q'$. The formula then is $\dfrac{1}{\rho}-\dfrac{\cos\phi}{R}-\dfrac{\sin\phi}{S}=V(p'q''-p''q')+\dfrac{\frac{1}{2}}{V}(E\dot{G}-\dot{E}G)$,

where the values $\dfrac{1}{R}$ and $\dfrac{1}{S}$ are now $=\dfrac{\frac{1}{2}G_1}{G\sqrt{E}}$ and $\dfrac{-\frac{1}{2}E_2}{E\sqrt{G}}$, respectively. But we have moreover $\phi=\tan^{-1}\dfrac{p'\sqrt{E}}{q'\sqrt{G}}$, and thence

$$\phi' = q'\sqrt{G}\left(p''\sqrt{E}+\frac{\tfrac{1}{2}p'\dot{E}}{\sqrt{E}}\right)-p\sqrt{E}\left(q''\sqrt{G}+\frac{\tfrac{1}{2}q'\dot{G}}{\sqrt{G}}\right),$$

$$= -V(p'q''-p''q')-\frac{\tfrac{1}{2}}{V}p'q'(E\dot{G}-\dot{E}G);$$

or the formula finally is

$$\frac{1}{\rho}-\frac{\cos\phi}{R}-\frac{\sin\phi}{S}+\phi'=0,$$

which is Liouville's formula referred to at the beginning of the present paper. It will be recollected that ϕ' is the differential coefficient $\frac{d\phi}{ds}$ with respect to the arc s of the curve.

Addition.—Since the foregoing paper was written, I have succeeded in obtaining a like interpretation of the term

$$V(p'q''-p''q')+\frac{\tfrac{1}{2}}{V}p'q'(Lp'+Mq'),$$

which belongs to the general case. I find that these terms are, in fact, $=-\dot{\phi}+\omega_1 p'$; or, what is the same thing (since $\omega=\phi+\theta$ and therefore $\omega_1 p'+\omega_2 q'=\dot{\phi}+\dot{\theta}$), are $=\dot{\theta}-\omega_2 q'$. It will be recollected that ϕ is the inclination of the curve to the curve $q=c$, which passes through a given point of the curve, $\dot{\phi}$ is the variation of ϕ corresponding to the passage to the consecutive point of the curve, viz., $\phi+\dot{\phi}\,ds$ is the inclination at this consecutive point to the curve $q=c+dc$, which passes through the consecutive point; ω is the inclination to each other of the curves $p=b$, $q=c$ which pass through the given point of the curve, ω_1 the variation corresponding to the passage along the curve $q=c$, viz., $\omega+\omega_1 ds$ is the inclination to each other of the curves $p=b+db$, $q=c$; and the like as regards $\dot{\theta}$ and ω_2.

For the demonstration, we have, as above,

$$\phi=\tan^{-1}\frac{Vp'}{Fp'+Gq'},\quad \omega=\tan^{-1}\frac{V}{F},\quad \text{where}\quad V=\sqrt{EG-F^2};$$

and moreover $Ep'^2+2Fp'q'+Gq'^2=1$. In virtue of this last equation, $V^2p'^2+(Fp'+Gq')^2=G$; and we have

$$\dot{\phi}=-V(p'q''-p''q')+\frac{1}{G}\square,$$

where $$\square=(Fp'+Gq')p'\dot{V}-Vp'(\dot{F}p'+\dot{G}q');$$

or, since $V^2 = EG - F^2$, and thence $2V\dot{V} = G\dot{E} - 2F\dot{F} + E\dot{G}$, we have

$$\square = \frac{\frac{1}{2}p'}{V}\{(Fp'+Gq')(G\dot{E}-2F\dot{F}+E\dot{G})-2(EG-F^2)(\dot{F}p'+\dot{G}q')\}.$$

Substituting herein for $\dot{E}$, $\dot{F}$, $\dot{G}$ their values $E_1p'+E_2q'$, $F_1p'+F_2q'$, $G_1p'+G_2q'$, the term in { } becomes

$$= Ip'^2 + Jp'q' + Kq'^2,$$

where $I = FGE_1 - 2EGF_1 + EFG_1$,

$$J = G^2E_1 - 2FGF_1 + (-EG+2F^2)\,G_1 + FGE_2 - 2EGF_2 + EFG_2,$$
$$K = G^2E_2 - 2FGF_2 + (-EG+2F^2)\,G_2.$$

But from the equation $\omega = \tan^{-1}\dfrac{V}{F}$, differentiating in regard to p,

we obtain $\omega_1 = \dfrac{\frac{1}{2}}{EGV}(FG\dot{E} - 2EG\dot{F} + EF\dot{G}) = \dfrac{\frac{1}{2}}{EGV}I$;

or, for p writing

$$p'(Ep'^2 + 2Fp'q' + Gq'^2), \quad = Ep'\left(p'^2 + 2\frac{F}{E}p'q' + \frac{G}{E}q'^2\right),$$

we have
$$\dot{\phi} - \omega_1 p' = -V(p'q''-p''q') + \frac{\frac{1}{2}p'}{GV}(Ip'^2 + Jp'q' + Kq'^2)$$
$$- \frac{\frac{1}{2}p'}{GV}I\left(p'^2 + 2\frac{F}{E}p'q' + \frac{G}{E}q'^2\right).$$

The terms in p'^3 destroy each other, and the form thus is

$$\dot{\phi} - \omega_1 p' = -V(p'q''-p''q') - \frac{\frac{1}{2}}{V}p'q'(Lp'+Mq'),$$

where
$$L = -\frac{J}{G} + \frac{2IF}{GE},$$
$$M = -\frac{K}{G} + \frac{I}{E};$$

and, upon substituting herein for I, J, K their values, we find

$$L = -GE_1 + EG_1 + \frac{2F^2E_1}{F} - 2FF_1 - FE_2 + 2EF_2 - \frac{EFG_2}{G},$$

$$M = -GE_2 + EG_2 - \frac{2F^2G_2}{G} + 2FF_2 + FG_1 - 2GF_1 + \frac{GFE_1}{E};$$

viz., these are the values denoted above by the same letters L, M. The final result thus is

$$\frac{1}{\rho} - \frac{q'\sqrt{G}}{R} - \frac{p'\sqrt{E}}{S} = -\dot{\phi} + \omega_1 p',$$
$$= \dot{\theta} - \omega_2 q',$$

where the meanings of the symbols have been already explained. A formula substantially equivalent to this, but in a different (and scarcely properly explained) notation, is given, "Aoust, Théorie des coordonnées curvilignes quelconques," *Annali di Matem.*, t. ii. (1868), pp. 39—64; and I was, in fact, led thereby to the foregoing further investigation.

As to the definition of the radius of geodesic curvature, I remark that, for a curve on a given surface, if PQ be an infinitesimal arc of the curve, then if from Q we let fall the perpendicular QM on the tangent plane at P (the point M being thus a point on the tangent PT of the curve), and if from M, in the tangent plane and at right angles to the tangent, we draw MN to meet the osculating plane of the curve in N, then MN is in fact equal to the infinitesimal arc QQ' mentioned near the beginning of the present paper, and the radius of geodesic curvature ρ is thus a length such that $2\rho\,.\,MN = \overline{PQ}^2$.

On certain Tetrahedra specially related to four Spheres meeting in a Point. By Samuel Roberts, F.R.S.

[*Read April 14th*, 1881.]

At our last meeting I mentioned an elementary theorem relating to a tetrahedron. Namely, if on each of the edges we take an arbitrary point, and describe a sphere through each vertex, and the three arbitrary points taken on the three adjacent edges, the four spheres meet in a point.

In this shape, the result is easily shewn by means of inversion. The needful preliminaries are as follows :—

(a) The analogue in plane space is this,—If an arbitrary point be taken on each side of a triangle, the three circles passing each through a vertex, and the two arbitrary points on adjacent sides, intersect in one point. This is a known result.

If we invert with regard to any point outside the plane, we get the following theorem :—

(b) If three circles on a sphere meet in a point forming by their intersections two and two together a triangular figure whose sides are circular, then, if we take any point on each of the circles and draw another circle through each simple intersection and the two arbitrary points taken on the circles to which the intersection belongs, these three circles last drawn meet in a point. This is the analogue on the sphere of the plane theorem. The resulting figure is, it will be ob-

served, perfectly symmetrical, consisting of six circles intersecting three together in eight points, and two together in six points.

Now, take a tetrahedron $ABCD$, and the arbitrary points a, a', a'', b, c, d on each of the edges. Suppose a sphere B passes through B, a, c, d; a sphere C through C, a', b, d; and a sphere D through D, a'', b, c.

Let the circle $rcpK$, meeting the side ADB in r and the side BCD in p, be the intersection of the spheres B and D. Similarly, let the circle $qbpK$, meeting the side ACD in q and the side BCD in p, be the intersection of the spheres C and D. Then K, p are the triple-intersections of the spheres B, C, D. If now a sphere be described through $Aaa'a''$, the points r, q, and a corresponding point s in the side ABC, become triple points of intersection of the spheres (A, B, D), (A, C, D), and (A, B, C) respectively. Then, by the analogue (b), the three circles (ars), $(a''rq)$, $(a'qs)$ meet in a point which must be the point K, and lies on the sphere A; that is to say, the four spheres A, B, C, D meet in a point.

It is perhaps worth while to regard the theorem from another point of view, taking as our data four spheres intersecting in a point. I had not worked out the question in this form when I presented the theorem which now becomes porismatic as in the corresponding plane case.

Consider the section of the three spheres B, C, D by any plane BCD, passing through p, a triple intersection of the spheres. Let BCD be any triangle so drawn that each side passes through an intersection of two of the section circles, and each vertex is on the section circle passing through the intersections on the adjacent sides.

Through BC, CD, DB respectively describe planes passing also through s, q, r respectively, the three other triple intersections of the spheres, s being the intersection of the spheres A, B, C; q that of the spheres A, C, D; and r that of the spheres A, B, D. Suppose that these planes meet in A'. And, as before, let the spheres B, C, D meet BA', CA', DA' in a, a', a'' respectively. Then the sphere through A', a, a', a'' passes through K, the other triple-intersection of the spheres B, C, D, and through q, r, s. This sphere therefore remains the same when the triangle BCD is porismatically varied, and is, in fact, the fourth given sphere A.

The series of triangles is singly infinite, and we shall see that the locus of A' is not the sphere A, but a circle thereon. But BCD may be any plane through the point p, and a system of planes through a point is doubly infinite. Hence the series of tetrahedra completely taken is trebly infinite.

If we consider the plane BCD as fixed, but the base triangle BCD variable, the edge AB (for instance) meets the two circles $arsK$, $Bcpd$, and the straight line sd. Hence in its different positions AB forms a

system of generators of a hyperboloid of one sheet, opposite to *sd*, which is also a generator. Moreover, the circles *arsK*, *Bcdp* are circular sections of opposite systems, and the sphere *A* meets the hyperboloid in another circle through *A* parallel to the circle *Bcpd*.

In like manner, the simultaneous movements of *AC*, *AD* generate two other hyperboloids, and the circle through *A* parallel to the base plane is common to them all. For a given position of the triangle, the same point *A* is the intersection of the corresponding generators through *B*, *C*, *D* respectively.

This follows at once from the remark, that if in the plane *BCD*, or in any parallel plane, we take a circle, and from any point thereof draw lines parallel to *pB*, *pC*, *pD* respectively, and through their remaining intersections with the circle draw lines parallel to the corresponding sides of the triangle, these last lines meet in a point on the circle. This theorem (*c*) is obvious when the figure is drawn.

It is easy to frame a line model exemplifying the foregoing conclusion, if we take for our base any three circles meeting in a point, and in a parallel plane any other circle.*

If now we move the plane *BCD* about *p*, the movement of *AB* generates a system of hyperboloids having in common a fixed circular section *arsK* and a common generator *pK* (in fact, the radical axis of the spheres *B*, *C*, *D*) of the same system as *AB*. This series of hyperboloids is doubly infinite, linear in two parameters.

Suppose now the base *BCD* and the circles thereon are given, also the circle through *A*, parallel to the base and the generator *pK* common to three hyperboloids, obtained as above, indicated. It is plain that, if through *any* point *K* of *Kp*, we take three circular sections of the hyperboloids opposite to the sections in the base respectively, we have the same singly infinite series of tetrahedra, but different sets of spheres, the three corresponding to *B*, *C*, *D* having the same radical axis.

When the plane *BCD* is given, the maximum tetrahedron is that one whose base is a triangle having its sides respectively perpendicular to *pb*, *pc*, *pd*.

And for any plane through *p*, that is to say, for four given spheres, generally the maximum tetrahedron has its sides perpendicular to the lines drawn from the quadruple intersection to the triple intersections; or, what amounts really to the same thing, the vertices of the tetrahedron are determined by the extremities of the right lines drawn through the quadruple intersection and the centres of the four spheres.

The minimum tetrahedron is represented by the line *Kp* common to the series of hyperboloids.

---

* The author exhibited a model of this kind.

The plane analogue (a) is immediately extended by general projection to the system of three conics having three triple intersections,* and it is natural to infer a similar generalization of the solid theorem. We have to consider the system of four conicoids, having a common plane section, a common quadruple intersection, and four triple intersections. With the assistance of the system of hyperboloids, a proof of the spherical case can be established which admits of immediate extension to the generalised form.

Thus (in outline) the four spheres being given, and any plane BCD through p a triple intersection, construct the hyperboloid generated by lines meeting the circles $Bcdp$, Krs, and the line Kp, and the other two hyperboloids similarly related to the circles $Cbdp$, $Dbcp$, &c. These three hyperboloids intersect again in a circle through A parallel to the plane BCD. The spheres on which this circle and the opposite sections through K lie, coalesce in one sphere determined by the point K.

For the generalised case, we have, instead of circles, conics meeting one and the same conic on a given plane, namely, the common plane section of the four conicoids; instead of opposite circular sections of the hyperboloids, we have conics passing through the points (α, β), (γ, δ) respectively, α, β, γ, δ denoting the four points in which the hyperboloid in question meets the common conic. Moreover, the theorem (c) is similarly extended by general projection. The reasoning in the case of spheres can now be immediately transferred to the generalised system of conicoids.

Observe that the conics in the base plane, and the conic through the vertex corresponding to the circle through A parallel to the base, are not parallel for finite positions of the common plane section.

The extended result can be also shewn by the theory of homologous figures in space of three dimensions.

Historical Note on Dr. Graves's Theorem on Confocal Conics. *By* Samuel Roberts, F.R.S.

[*Read April* 14*th*, 1881.]

Dr. Graves's theorem for plane conics (1841) is as follows:—If two tangents be drawn to an ellipse from any point of a confocal ellipse,

---

* Or, if two conics α, β intersect in the points a, b, c, d, and through d we draw a transversal meeting α in k, and β in l, and if p is a fixed point on α and q a fixed point on β, we see at once by anharmonic ratios, taking four positions, that the intersection of the lines kp, lq moves on a conic through a, b, c, p, q.

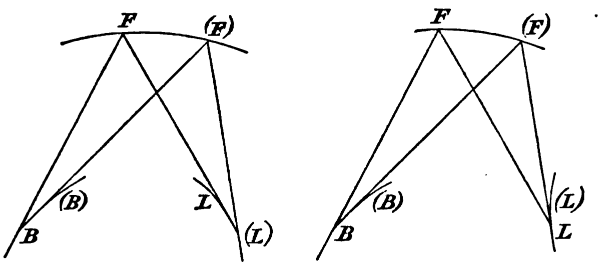
the excess of the sum of these two tangents over the arc intercepted between them, is constant.

Dr. Salmon states, as a corollary to this, that the same theorem will be true for any two curves which possess the property that two tangents TP, PQ to the inner one always make equal angles with the tangent TT' to the outer.

In a letter dated 3rd Jan., 1704, addressed to John Bernoulli, Leibnitz gives the general theorem in this form:

He supposes BF, $(B)\,(F)$, to represent rays reflected by a mirror $F\,(F)$, so that, by the rule of Caustics, the points L, (L) correspond to B, (B) respectively. The reflected rays envelop a curved line $L\,(L)$ corresponding to the curved line $B\,(B)$. If $B\,(B)$, $L\,(L)$ are given, the curve $F\,(F)$ is described by a pencil point stretching a thread BFL enveloping the two curves. This process of tranforming $B\,(B)$ into $L\,(L)$ is called "coëvolutio." Then

$$BFL + L\,.\,(L) = B\,.\,(B) + (B)\,(F)\,(L).$$



Dr. Graves's theorem for plane conics is evidently a particular case. The theory of coëvolution is given as affording a solution of the problem proposed by Bernoulli (on the suggestion of another however), namely, to transform a given curve into an infinity of other curves of the same length.

Leibnitz points out that the curve $F\,(F)$ touches all the ellipses whose foci are the corresponding points B, L, and whose focal sum is BFL. But he also remarks that, if we suppose the reflected rays to be convergent behind the mirror, this curve is a line cutting the ellipses orthogonally. Mr. MacCullagh's theorem corresponds to this, the curve from which the tangents are drawn being a confocal hyperbola.

Leibnitz also suggests the case of refraction generally instead of reflection. This theory of coëvolution is several times referred to in the correspondence between Leibnitz and Bernoulli, and is mentioned by the latter in his paper on "Motus Reptorius" (*Act. Erud. Lips.*, 1703, Opera, Tom. i., No. lxxii., p. 406); but it does not appear to have been worked out.

It should be remarked that Dr. Graves gave the theorem as of spherical conics. So also did Chasles (1843). The theory of Chasles is very complete on arcs of conics whose difference is rectifiable.

On Operative Symbols in the Differential Calculus.

By M. W. Crofton, F.R.S.

[*Read April* 14*th*, 1881.]

1. The following paper contains several results in the theory of operative symbols as employed in the differential calculus. To claim them all as new would be, in these days, a somewhat hazardous pretension; the author is, besides, fully sensible of the fragmentary and unfinished shape in which a few detached results are here presented, which evidently form but indications of a great extension of the infinitesimal calculus.

2. The present paper is confined to the consideration of symbols of the form $f(x, D)$; where $D = \frac{d}{dx}$. If η be any such symbol, it is well to remember that it is commutative with any function of itself; in fact, that

$$\phi(\eta)\,\psi(\eta) \equiv \psi(\eta)\,\phi(\eta).$$

In treating of these symbols, an operand will be usually understood when not expressed. A symbol may be *isolated*; *i.e.*, considered independently of any operand; it is important to remember that this is different from the same symbol operating on unity. To illustrate this distinction, and the notation employed, let us consider the symbol xD, as operated on by D^{-1}, in the following cases:—

$$\begin{array}{ll} 1.\ D^{-1}\ xD & \equiv x - D^{-1}, \\ 2.\ D^{-1}\,|xD| & \equiv \tfrac{1}{2}x^2D + \phi(D), \\ 3.\ D^{-1}\ xD\ .1 = C, & \\ 4.\ D^{-1}\,|xD|\,.1 = C. & \end{array}$$

In (1) and (2), an operand is understood; but in (2) the operation D^{-1} is limited by the bars $|\,|$ to the symbol xD.

3. In dealing with these symbols, great care is required not to misapply, through habit, the ordinary rules of algebra and of differentiation, &c. For instance, if we require to differentiate the symbol

$$u = \overline{xD}|^{2}$$

with regard to x, it is incorrect to give

$$\frac{du}{dx} = 2\overline{xD}|\,D,$$

for we must remember that $u = xDxD$. We have here to go back to the first principles of the differential calculus, and consider the increment of u produced by a small increment of x; and we find

$$\frac{du}{dx} = DxD + xDD.$$

As we have the identity $\qquad Dx \equiv 1 + xD,$

this becomes $\qquad \dfrac{du}{dx} = 2\overline{xD}|\,D + D.$

To give rules for the differentiation of such symbols in general, is more than the present state of knowledge admits of; we shall have occasion to treat a few isolated cases in what follows.

4. The two following formulas are of frequent application. If η be

any such operator, $\qquad D\,|\,\eta\,| = \dfrac{d\eta}{dx} \equiv D\eta - \eta D \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$

$$\frac{d\eta}{dD} \equiv \eta x - x\eta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The first follows easily by considering that

$$D\eta X = \frac{d\eta}{dx}X + \eta\frac{dX}{dx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3);$$

the second from $\eta x X = f(x, D)\,xX = f(x, D_0 + D_1)\,xX,$

where D_0 is null as regards X, and D_1 as regards x, so that

$$\eta x X = x\eta X + \frac{d\eta}{dD}X.$$

As an example of how these formulas may sometimes be made use of, let us propose to differentiate

$$u - \overline{xD}|^r.$$

$$\begin{aligned}\frac{du}{dx} &= D\overline{xD}|^r - \overline{xD}|^r D \\ &= \overline{Dx}|^r D - \overline{xD}|^r D \\ &= (\overline{1+xD}|^r - \overline{xD}|^r)\,D.\end{aligned}$$

If we put $xD = \zeta$, this may be written

$$\frac{du}{dx} = \Delta\zeta^r D,$$

where Δ operates only on ζ; therefore

$$\frac{d}{dx}\overline{xD}|^{r} = \left(r.\overline{xD}|^{r-1}+\frac{r(r-1)}{1.2}\overline{xD}|^{r-2}+\&c.\right)D \quad\text{.........(4)},$$

also
$$\frac{d}{dx}f(xD) \equiv \Delta f(\zeta)D \equiv f'(xD)D+\frac{1}{1.2}f''(xD)D+\&c.\text{....(5)},$$

thus
$$\frac{d}{dx}e^{xD} \equiv D|e^{xD}| \equiv (e-1)e^{xD}D \quad\text{............................(6)}.$$

5. In a paper in the "Quarterly Journal of Mathematics," Oct., 1879, I have given the theorem

$$\phi\left(\frac{d}{dD}\right)f(D) \equiv f(D)\phi(x-\dot{x})\text{....................(7)},$$

a series equivalent to which had previously been given by Prof. Donkin. In the second member, $\dot{x}$ is a letter as regards which D is null; it being replaced by x at the conclusion of the operations. This theorem can be generalized, and holds when $f(D)$ involves x as well as D; so that, if η be any operator,

$$\phi\left(\frac{d}{dD}\right)\eta \equiv \eta\phi(x-\dot{x}) \quad\text{......................(8)},$$

for, from (2),

$$\left(\frac{d}{dD}\right)^2\eta = (\eta x-x\eta)x-x(\eta x-x\eta) = \eta x^2-2x\eta x+x^2\eta;$$

therefore
$$\left(\frac{d}{dD}\right)^2\eta \equiv \eta(x-\dot{x})^2,$$

and this can easily be extended to any function ϕ.

From (8),
$$e^{\frac{d}{dD}}\eta \equiv e^{-x}\eta e^{x},$$

hence
$$f(x, D+1) \equiv e^{-x}f(x, D)e^{x}\text{......................(9)},$$

an extension of the well-known theorem as to $f(D)$; it is easily extended to $f(x, D+r)$.

For example,

$$a^{x(D+r)}F(x) = e^{-rx}a^{xD}e^{rx}F(x) = e^{-rx}e^{rax}F(ax),$$

therefore
$$a^{x(D+r)}F(x) = e^{rx(a-1)}F(ax)\text{....................(10)},$$

as it is known that the effect of the operator a^{xD} is to change x into ax in the subject.

6. An important principle in this theory is, that we may change D in an operator into D_0+D; thus

$$\eta \equiv f(x, D) \equiv f(x, D_0+D) \quad\text{....................(11)},$$

understanding that, *in the second member, D is null as regards the operator, and D_0 null as regards the operand.*

Thus we may put, in this sense, instead of the operator $\eta = \overline{xD}|^2$,

$$\eta = \overline{x(D_0+D)}|^2,$$

therefore $$\eta = \overline{xD_0}|^2 + xD_0xD + xDxD_0 + \overline{xD}|^2.$$

Here the first and third terms vanish, the second becomes xD, and the fourth x^2D^2, so that

$$\overline{xD}|^2 = xDxD = xD + x^2D^2,$$

as is easily verified.

By expanding the second member of (11) according to powers of D, and suppressing the suffix 0; using f also for shortness instead of $f(x, D)$,

$$f(x, D)\,X \equiv f(x, D)\Big|\,X + \frac{df}{dD}\Big|\,DX + \frac{1}{1\,.\,2}\,\frac{d^2f}{dD^2}\Big|\,D^2X + \&\text{c.} \quad\text{.......(12)},$$

the bars limiting the effect of the D preceding them, the subject unity being understood before each bar.

To give an application of this theorem, let us consider (putting $\zeta = xD$), $$u = f(xD)\,F(x) = f(\zeta)\,F(x).$$

As in Art. 4, we can easily show that

$$\frac{d}{dD}f(xD) = x\,\{f(xD+1) - f(xD)\} = x\Delta f(\zeta) \quad\text{.........(13)},$$

where Δ operates only on ζ. Hence

$$f(\zeta)\,F(x) = \Big\{f(\zeta)\,\Big| + x\Delta f(\zeta)\,\Big|\,D + \frac{x^2}{1\,.\,2}\Delta^2 f(\zeta)\,\Big|\,D^2 + \dots\Big\}\,F(x)$$
$$= e^{\dot{x}\Delta D} f(\zeta)\,|\,F(x) \quad\text{..........................(14)},$$

where $\dot{x}$ is null as to all the operators, Δ affects only ζ, D affects only $F(x)$, and ζ is null as to $F(x)$, as is indicated by the bar.

Therefore $$u = f(\zeta)\,F(x) = F(x + \dot{x}\Delta)\,f(\zeta)\,|,$$

or $$u = F[\dot{x}(1+\Delta)]\,f(\zeta)\,| = F\big(\dot{x}e^{\frac{d}{d\zeta}}\big)\,f(\zeta)\,.\,1 \quad\text{...........(15)}.$$

Thus, if we have to expand $u = f(xD)\,e^x$ according to powers of x,

$$u = \exp.\,\big(\dot{x}e^{\frac{d}{d\zeta}}\big)\,f(\zeta)\,.\,1$$
$$= f(\zeta) + xf(\zeta+1) + \frac{x^2}{1\,.\,2}f(\zeta+2) + \dots,$$

the right member operating on 1. But

$$f(xD+r)\,.\,1 = f(r),$$

therefore $$f(xD)\,e^x = f(0) + xf(1) + \frac{x^2}{1\,.\,2} f(2) + \&c. \quad \text{.........(16)},$$

easily verified by first expanding e^x. Again, if

$$u = f(xD)\,(\log x)^r,$$
$$u = \left(\log \dot{x} + \frac{d}{d\zeta}\right)^r f(\zeta)\,.\,1,$$

therefore

$$u = \log^r x f(0) + r \log^{r-1} x\,.\,f'(0) + \frac{r(r-1)}{1\,.\,2} \log^{r-2} x\,.\,f''(0) + \&c.\text{....(17)}.$$

Of course the chief difficulty in applying such a formula as (12) is the absence of rules for differentiating symbols with regard to the x or D which they contain.

7. Lagrange's Theorem.—We may here give a proof of Lagrange's theorem by this method; let us propose to find the value of

$$u = e^{D\cdot\phi} F(x),$$

where, for shortness, ϕ is put for $\phi(x)$; a point is placed between D and ϕ (though dropped for shortness in what follows) to remind us that $D\phi$ is not here an indivisible symbol, but that in fact we define

$$e^{D\cdot\phi} \equiv 1 + D\phi + \frac{D^2\phi^2}{1\,.\,2} + \frac{D^3\phi^3}{1\,.\,2\,.\,3} + \&c. \quad \text{..............(18)}.$$

As in (11), we may write here $D_0 + D$ for D, and we easily see, from (18), that $$u = e^{(D_0+D)\phi} F(x) = e^{D_0\phi} e^{D\phi} F(x),$$

where D only affects $F(x)$, being null as regards $\phi(x)$; hence

$$u = e^{D_0\phi} F[\dot{x} + \phi(x)] = e^{D\phi} F[\dot{x} + \phi(x)],$$

dropping the suffix 0, and marking x with a point to denote that D is null as regards it.

Thus $$u \equiv e^{D\phi} F(x) \equiv e^{D\phi} F[\dot{x} + \phi(x)],$$

so that we may replace x by $\dot{x} + \phi(x)$ without change; but, if so, we may repeat the process, and put

$$u = e^{D\phi} F\{\dot{x} + \phi(\dot{x} + \phi x)\},$$

and so on *ad infinitum*. Hence, putting

$$z = \dot{x} + \phi\{\dot{x} + \phi(\dot{x} + \ldots)\} = \dot{x} + \phi(z),$$
$$u = e^{D\phi} F(z) = F(z)\, e^{D\phi}\,.\,1 \quad \text{....................(19)},$$

as D is null as to z. To determine $e^{D\phi}\,.\,1$, it is easy to see, from (18), that $$e^{D\phi}\,.\,1 = 1 + e^{D\phi}\phi'(x) = 1 + \phi'(z)\, e^{D\phi}\,.\,1,$$

therefore $$e^{D\phi}.1 = \frac{1}{1-\phi'(z)} = \frac{dz}{dx}.$$

Thus we have proved $u = e^{D\phi}F(x) = F(z)\dfrac{dz}{dx}$ ....(20),

where $$z = x+\phi(z) \text{ ....(21);}$$

put F' for F, and we have

$$\frac{d}{dx}F(z) = e^{D\phi}F'(x),$$

or $$F(z) = F(x)+\phi\,.\,F'(x)+\frac{1}{1\,.\,2}D\phi^2\,.\,F'(x)+\dots$$

which is Lagrange's theorem.

8. Since $\overline{x^2D}|^n x^r = r(r+1)\dots(r+n-1)\,x^{r+n}$,

it is easy to shew by expanding the exponential that

$$e^{x^2D}x^r = \left(\frac{x}{1-x}\right)^r \text{ ....(22),*}$$

and thence that $$e^{x^2D}f(x) \equiv f\left(\frac{x}{1-x}\right) \text{ ....(23);}$$

and, in a similar way, it can be proved that in general

$$\exp.(ax^{r+1}D)f(x) \equiv f\left\{\frac{x}{(1-rax^r)^{\frac{1}{r}}}\right\} \text{ ....(24).}$$

This may be shewn to include the known cases, when $r=0$,

$$e^{axD}f(x) = f(e^a x),$$

and Taylor's theorem, when $r=-1$. We have also

$$\exp.(ax^{-1}D)f(x) \equiv f(\sqrt{x^2+2a}) \text{..................(25);}$$

thus the expansion of this by powers of a is

$$f(\sqrt{x^2+2a}) = f(x)+a\,\overline{x^{-1}D}|\,f(x)+\frac{a^2}{1\,.\,2}\overline{x^{-1}D}|^2 f(x)+\dots$$

By considering $f(x)$ as consisting of a series of powers of e^x, it is not difficult to shew also that

$$\exp.(e^{ax}D)f(x) \equiv f\left\{x-\frac{1}{a}\log(1-ae^{ax})\right\} \text{ ....(26).}$$

9. The above however are only cases of a general theorem, somewhat resembling Lagrange's, which was first arrived at by induction from such cases as the above, but which, once stated, can be easily proved as follows.

* N.B.—x^2D is here an indivisible symbol, its square being x^2Dx^2D, not x^4D^2, &c. The same remark applies to $e^{\phi D}$, $e^{D\phi}$, &c., in what follows; so that the series (18) now ceases to be employed.

If z, x are related by the equation

$$\psi(z) = 1 + \psi(x) \quad \text{.........................(27)},$$

it is required to express any function $F(z)$ of z in terms of x.

The result is very simple, viz., putting for convenience of notation

$$\phi = \phi(x) = \frac{1}{\psi'(x)}, \quad i.e., \ \psi(x) = D^{-1}\phi^{-1} \quad \text{............(28)},$$

we shall have

$$F(z) \equiv e^{\phi D} F(x) \quad \text{...........................(29)}.$$

The simplest proof is probably to observe first that (28)

$$\phi D \{\psi(x)\}^n = n \{\psi(x)\}^{n-1} = n\psi^{n-1},$$

putting ψ for $\psi(x)$; hence $\phi Df(\psi) = f'(\psi)$,

and hence

$$F(\phi D) f\{\psi(x)\} \equiv \Big]_x^{\psi} F(D) f(x) \quad \text{.................(30)},$$

Sarrus' *sign of substitution* being used to show that $\psi(x)$ is finally put for x. A case of (30) is

$$e^{\phi D} f(\psi) = \Big]_x^{\psi} e^D f(x) = f(\psi + 1),$$

that is, by (27),

$$e^{\phi D} f(\psi x) = f(\psi z),$$

or

$$e^{\phi D} F(x) = F(z),$$

a general theorem which includes the cases in Art. 9, and, of course, many others.

10. To give one or two more examples of the application of (29); let it be proposed—(1) to find an operator which *for all functions of x, shall change x into $ax + b$.*

In (27) we have to determine the function ψ from the equation

$$\psi(ax + b) \equiv 1 + \psi(x);$$

this is satisfied by

$$\psi(x) = \frac{1}{\log a} \log\left(x - \frac{b}{1-a}\right);$$

hence

$$a^{\left(x - \frac{b}{1-a}\right) D} F(x) \equiv F(ax + b) \quad \text{.................(31)},$$

From this we have

$$a^{xD} e^{bD} \equiv a^{\left(x - \frac{b}{1-a}\right) D}.$$

(2.)

$$\text{exp.}\left(\overline{\log x \,.\, D}\,\big|\right) F(x) = F(z) \quad \text{.................(32)},$$

where z is related to x by $li(z) = 1 + li(x)$ ....(33),

li denoting the logarithm-integral,

$$li(x) = \int_0 \frac{dx}{\log x}.$$

Hence, if z is related to x, as in (33), its value in terms of x is

$$z = x + \overline{\log x \,.\, D}|\, x + \frac{1}{1\,.\,2}\, \overline{\log x \,.\, D}|^2 x + \&c.$$

(3.) To find an operator which shall change x to x^r, in any function. The solution of the functional equation

$$\psi(x^r) \equiv 1 + \psi(x)$$

is
$$\psi(x) = \frac{\log \log x}{\log r} + C.$$

Hence (29)
$$r^{x \log x D} F(x) \equiv F(x^r) \quad \text{......................(34)},$$

$$\exp.\,(x \log x \,.\, D)\, F(x) = F(x^e) \quad \text{..................(35)}.$$

(4.) To find a function X such that, whatever be the function F,

$$F(e^x D)\, X \equiv F(a)\, X.$$

In (28) put $\phi(x) = e^x$, therefore $\psi(x) = -e^{-x}$; by (30),

$$F(e^x D) f(-e^{-x}) = \Big]_x^{-e^{-x}} F(D) f(x).$$

Now, if $f(x) = e^{ax}$, this becomes $F(a) f(-e^{-x})$; hence the required function is
$$X = \exp.\,(-ae^{-x}).$$

11. If we put for shortness ϕ for $\phi(x)$, ψ for $\psi(x)$, these being any two functions, and
$$u = D^{-1} \frac{\psi}{\phi}\Big|, \quad \text{or} \quad \int \frac{\psi(x)}{\phi(x)}\, dx,$$

then for any operand
$$\phi D e^u = \phi \left(e^u D + e^u \frac{\psi}{\phi} \right)$$
$$= e^u (\phi D + \psi),$$

and, by repeating the operation,

$$f(\phi D)\, e^u = e^u f(\phi D + \psi);$$

so that in general, whatever be the operand,

$$f(\phi D + \psi) \equiv \exp.\left(-D^{-1} \frac{\psi}{\phi} \right) f(\phi D)\, \exp.\left(D^{-1} \frac{\psi}{\phi} \right)^* \quad \text{...... (36}$$

Let $\psi(x) = \phi'(x)$, $f(\phi D + \phi') = \phi^{-1} f(\phi D)\, \phi$;

but
$$\phi D + \phi' \equiv D\phi,$$

hence
$$\phi f(D\phi) = f(\phi D)\, \phi \quad \text{..........................(37)}$$

this may be applied to the results in Arts. 8, 9. Thus, from (23),

$$e^{Dx^2} f(x) = x^{-2} e^{x^2 D} x^2 f(x) = \frac{1}{(1-x)^2} f\left(\frac{x}{1-x} \right).$$

* N.B.—In the exponentials, D^{-1} only affects the function $\frac{\psi}{\phi}$.

Again, from (29),

$$e^{D\psi} F(x) = \frac{1}{\phi(x)} e^{\psi D} \phi(x) F(x) = \frac{\phi(z)}{\phi(x)} F(z).$$

But $\psi(z) = 1 + \psi(x)$, where $\psi(x) = D^{-1}\phi^{-1}$, therefore $\psi'(x) = \dfrac{1}{\phi}$,

therefore $$\frac{dz}{dx} = \frac{\phi(z)}{\phi(x)},$$

therefore $$e^{D\psi} F(x) = \frac{dz}{dx} F(z) \quad \text{.......................(38).}$$

This would have followed at once also from the second formula in

$$f(D\phi) \equiv \phi^{-1} f(\phi D)\, \phi \equiv D f(\phi D)\, D^{-1} \quad \text{...............(39),}$$

which can be shewn by expansion; as well as by (36), as regards the first result. The second also follows from the first, by the remark in Art. 2.

12. If, in (36), we put $\phi(x)\, a'(x)$ instead of $\psi(x)$, $a(x)$ being any function; calling it a, and $\phi(x)$, ϕ,

$$f\{\phi(D + a')\} = e^{-a} f(\phi D)\, e^{a} \quad \text{.........................(40).}$$

Thus, in the theorem in (29), let D receive as an increment any function $a'(x)$, $$\exp.\{\phi[D + a'(x)]\} F(x) = e^{-a(x)} e^{a(z)} F(z) \quad \text{.........(41),}$$
z being related to x by equation (27).

Suppose, for example, $\phi(x) = 1$, then $z = x + 1$,

$$e^{D + a'(x)} F(x) = e^{-a(x)} e^{a(x+1)} F(x+1),$$

$a(x)$ being any function; this is easily deducible from known theorems.

Many results might be made to follow from (41); *e.g.*, to find the value of $$U = \exp.(x^2 D + x^3) F(x).$$

Here $a'(x) = x$, $a(x) = \frac{1}{2}x^2$; hence (23)

$$U = e^{-\frac{1}{2}x^2} e^{x^2 D} e^{\frac{1}{2}x^2} F(x) = \exp.(-\tfrac{1}{2}x^2) \,.\, \exp.\left(\frac{\frac{1}{2}x^2}{(1-x)^2}\right) F\left(\frac{x}{1-x}\right) \text{...(42).}$$

13. If, in (36), we put $\psi(x) = 1$, and for f the exponential, and

$$D^{-1}\phi^{-1} = \beta(x) = \beta,$$

$$e^{1+\phi D} \equiv e\, e^{\phi D} \equiv e^{-\beta} e^{\phi D} e^{\beta}.$$

Hence $$e^{\phi D} e^{\beta} = e^{1+\beta} e^{\phi D},$$

therefore $$e^{\phi D} e^{r\beta} = e^{r + r\beta} e^{\phi D},$$

therefore $$e^{\phi D} f(e^{\beta}) = f(e^{1+\beta})\, e^{\phi D}.$$

If we now put $$1 + \beta(x) = \beta(z),$$

and take unity as operand, we shall easily find

$$e^{\phi D} F\{\beta(x)\} = F\{\beta(z)\},$$

therefore $$e^{\phi D} F(x) = F(z),$$

thus proving the theorem in (29) by a different process.

14. If we put $\zeta = D\phi$, where $\phi = \phi(x)$, and if u, v are functions of x,

$$\zeta uv = vD\phi u + u\phi Dv$$
$$= v\zeta u + uD^{-1}\zeta Dv,$$

therefore $$\zeta uv = (\zeta' + D_1^{-1}\zeta_1 D_1)\,uv \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(43),$$

provided ζ' is null as to v, and D_1, ζ_1 null as to u.

By repeating this operation on uv, it is easy to see that

$$\zeta^n uv = (\zeta' + D_1^{-1}\zeta_1 D)^n\, uv, \text{ or}$$

$$\overline{D\phi}\rceil^n uv = D_1^{-1}\left\{v'\zeta^n u + n\zeta v'.\zeta^{n-1}u + \frac{n(n-1)}{1.2}\zeta^2 v'.\zeta^{n-2}u + \ldots\right\}\ldots(44),$$

where $v' = Dv$; D_1^{-1} refers only to v', and the points indicate multiplication.

It is also evident, from (43), that

$$e^{\zeta} uv = e^{\zeta} u \,.\, D^{-1} e^{\zeta} Dv.$$

Another form may be given to (44), by putting

$$\eta = \phi D,$$

η only operating on v, ζ only on u; we shall then have

$$D\phi uv = (\zeta + \eta)\,uv,$$

and $$\overline{D\phi}\rceil^n uv = v\zeta^n u + n\eta v\,.\,\zeta^{n-1}u + \frac{n(n-1)}{1.2}\eta^2 v'\,.\,\zeta^{n-2}u + \ldots \quad\ldots(45).$$

As an example, put $v = \exp.\,(D^{-1}\phi^{-1})$, then

$$\overline{D\phi}\rceil^n vu = v\,(1 + D\phi)^n u,$$

a case of (36).

15. Mr. Walker has given ("Proceedings, London Mathematical Society," April, 1880) a very remarkable formula for $D^n\phi^n uv$, where $\phi = \phi(x)$, and u, v are functions of x, viz.,

$$D^n\phi^n uv = vD^n\phi^n u + nv'\phi\,.\,D^{n-1}\phi^{n-1}u + \frac{n(n-1)}{1.2}D\phi^2 v'\,.\,D^{n-2}\phi^{n-2}u + \&c.$$

This may be put in the form

$$D^n\phi^n uv = D_1^{-1}\left\{v'\,.\,D^n\phi^n u + nD\phi v'.D^{n-1}\phi^{n-1}u \right.$$
$$\left. + \frac{n(n-1)}{1.2}D^2\phi^2 v'\,.\,D^{n-2}\phi^{n-2}u + \&c.\right\}\ldots\ldots(46),$$

being, as is very remarkable, exactly the same series as (44), except that $D^r\phi^r$ everywhere takes the place of $\overline{D\phi}\rceil^r$.

To establish this result by the method of symbols, let U be any operator, being a function of x, D, and let us define the symbolic operator ζ to be $\qquad \zeta U| \equiv DU\phi$ ....(47).

Hence $\qquad \zeta^r U| \equiv D^r U\phi^r.$

Let v be any function of x, then

$$DUv\phi = UDv\phi + DU|\phi v,$$

therefore $\qquad \zeta Uv = U\zeta v + \{D^{-1}\zeta\dot{D}U|\}\, v,$

where $\dot{D}$ is marked so, to indicate that the operation $\dot{D}$ is performed *before* the ζ which precedes.

This may be put $\qquad \zeta Uv = (\zeta_2 + D_1^{-1}\zeta_1\dot{D}_1)\, Uv,$

the suffix $_1$ denoting operations on U only, $_2$ on v only, the former being thus commutative with the latter. We shall have thus

$$\zeta^2 Uv = (\zeta_2^2 + 2\zeta_2 D_1^{-1}\zeta_1\dot{D}_1 + D_1^{-1}\zeta_1^2\dot{D}_1)\, Uv,$$

since $\qquad D^{-1}\zeta\dot{D}D^{-1}\zeta\dot{D}U| = D^{-1}\zeta\zeta\dot{D}U|.$

Hence, if for shortness we put $DU| = U'$,

$$\zeta^n Uv = D^n Uv\phi^n = D_1^{-1}\Big\{U'D^n\phi^n v + nDU'\phi|D^{n-1}\phi^{n-1}v$$
$$+\frac{n(n-1)}{1.2}D^2U'\phi^2|D^{n-2}\phi^{n-2}v + \&c.\Big\} \ldots\ldots(48),$$

which includes Mr. Walker's theorem (46), when $U = u$, a function of x (the letters u, v being interchanged). The bars in (48) limit the operation of the D, D^2, &c., which precede them, also of D_1^{-1} at the commencement, which is null as regards v.

16. As we have from last article, if u, v are functions of x,

$$f(\zeta)\, uv \equiv f(\zeta_2 + D_1^{-1}\zeta_1\dot{D}_1)\, uv,$$
$$e^{\zeta}uv = \exp.(D^{-1}\zeta D)\, u\,.\, e^{\zeta}v.$$

Suppose $v = 1$, $\qquad e^{\zeta}u = e^{\zeta}1\,.\exp.(D^{-1}\zeta D)\, u$ ....(49),

or $\qquad e^{\zeta}u = e^{\zeta}1\,.\,D^{-1}e^{\zeta}Du.$

Let $u = x$, $\qquad e^{\zeta}x = e^{\zeta}1\,.\,D^{-1}e^{\zeta}1;$

if we put $z = D^{-1}e^{\zeta}1$,

$$D^{-1}e^{\zeta}x = \tfrac{1}{2}z^2;$$

therefore $\qquad D^{-1}e^{\zeta}Dx^2 = z^2,$

and in like manner $\qquad D^{-1}e^{\zeta}Dx^r = z^r;$

therefore $\qquad D^{-1}e^{\zeta}DF(x) = F(z)$ ....(50),

where $$e_\zeta = 1 + D\phi + \frac{D^2\phi^2}{1.2} + \&c. \quad \text{.......................(51).}$$

This is Lagrange's theorem proved by a different process; for

$$z = x + \phi(z),$$

because, from (50), $\quad D^{-1}e^{\zeta}D\phi(x) = \phi(z),$

and it is easy to see, (51), that

$$e^{\zeta}1 = 1 + e^{\zeta}D\phi(x),$$

therefore $$z = D^{-1}e^{\zeta}1 = x + D^{-1}e^{\zeta}D\phi(x).$$

17. Mr. Hargreave has remarked (*Phil. Trans.*, 1848, see Boole's "Differential Equations," p. 455) that any relation between the symbols x and D will hold if we substitute D for x, and $-x$ for D. Thus Taylor's theorem gives us the relation

$$e^{hD}f(x) \equiv f(x+h)\, e^{hD},$$

any subject being understood. By the above substitution, we deduce the known relation $\quad f(D+h) \equiv e^{-hx} f(D)\, e^{hx}.$

Again, Lagrange's theorem may be stated

$$e^{D.\phi(x)}F(x) \equiv F(z)\, e^{D.\phi(z)} \quad \text{.....................(52),}$$

where we define $\; e^{D.\phi(x)} \equiv 1 + D\phi x + \frac{1}{1.2} D^2 (\phi x)^2 + \&c. \quad$ ....(53),

and $$z = x + \phi(z).$$

We deduce $$e^{-x.\phi(D)}F(D) \equiv F(\zeta)\, e^{-x.\phi(D)} \quad \text{..................(54),}$$

where $$\zeta \equiv D + \phi(\zeta) \quad \text{..............................(55).}$$

It may easily be shewn that the reciprocal or inverse of the operator $e^{D.\phi x}$, in (53), is not $e^{-D.\phi x}$, but $e^{-D.\phi z}$. Hence also

$$e^{-x.\phi D}\, e^{x.\phi\zeta} \equiv 1,$$

so that (54) gives $$F(\zeta) \equiv e^{-x.\phi D} F(D)\, e^{x.\phi\zeta} \quad \text{.....................(56).}$$

Suppose, *e.g.*, $$\phi(D) = hD,$$

then $$\zeta = \frac{D}{1-h},$$

therefore $$F\left(\frac{D}{1-h}\right) \equiv e^{-hx.D} F(D)\, e^{\frac{hx}{1-h}D}.$$

Again, if $\phi(D) = D^2$; by taking in (56), $F(D-D^2)$ for $F(D)$,

$$e^{-x.D^2} F(D-D^2) \equiv F(D)\, e^{-x.D^2} \quad \text{..................(57),}$$

which is easy to verify when e^{hx} is the subject.

If we take $\phi(D) = D - D^2$; then, from (55), $\zeta^2 = D$,

$$F.(D^{\frac{1}{2}}) \equiv e^{x.(D^2-D)}\, F(D)\, e^{x.(D^{\frac{1}{2}}-D)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (58).$$

18. Many of the results in this paper may be transformed in like manner. Thus from formula (23) we derive

$$F\left(\frac{D}{1-D}\right) \equiv e^{-D^2x}\, F(D)\, e^{D^2x},$$

or

$$^{*}F\left(\frac{D}{1-D}\right) \equiv e^{-xD^2}\, F(D)\, e^{xD^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (59),$$

where, we must remember, D^2x is treated as one symbol, and the definition (53), relating to Lagrange's theorem and its applications, no longer holds. In (59) let the subject be

$$e^{-xD^2}\, e^{hx},$$

$$F\left(\frac{D}{1-D}\right) e^{-xD^2} e^{hx} = F(h)\, e^{-xD^2} e^{hx}.$$

This is easy to verify, because

$$e^{kxD^2} e^{hx} = \exp.\left(\frac{hx}{1-hk}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (60).$$

It is perhaps worth noticing one method of deducing this theorem: by putting

$$u = e^{kxD^2} e^{hx},$$

we find, on differentiating,

$$\frac{du}{dk} = h^2 \frac{du}{dh},$$

therefore

$$u = \psi\left(\frac{h}{1-hk}\right);$$

but when $k = 0$,

$$u = \psi(h) = e^{hx}.$$

Since

$$\exp.\,(DaD^{-1}) \equiv D \exp.\,(a)\, D^{-1},$$

whatever be a, we deduce from (60)

$$e^{kDxD} e^{hx} = De^{kxD^2} h^{-1} e^{hx}$$

$$= \frac{1}{1-hk} \exp. \frac{hx}{1-hk} \ldots\ldots\ldots\ldots\ldots\ldots (61).$$

Again,

$$e^{kD^2x} e^{hx} = \frac{1}{(1-hk)^2} \exp. \frac{hx}{1-hk} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (62).$$

19. Since, by formula (34),

$$r^{(\log x)D} F(x) \equiv F(x^r)\, r^{(\log x)D}$$

we deduce

$$F(D^r) \equiv r^{-(\log D)x} F(D)\, r^{(\log D)x} \quad \ldots\ldots\ldots\ldots\ldots\ldots (63).$$

The full significance of this and other similar formulas, no doubt, can hardly be appreciated, till our present limited knowledge of the theory of these symbols be much extended.

---

* Because $e^{D^2x} = D^2 e^{xD^2} D^{-2}$; also for e^{-D^2x}.

On the General Equation of the Second Degree in Tetrahedral Coordinates. By Mr. Harry Hart.

[*Read April 14th*, 1881.]

1. Let the equation to the surface be

$$\phi(\alpha, \beta, \gamma, \delta) \equiv u_{11}\alpha^2 + u_{22}\beta^2 + u_{33}\gamma^2 + u_{44}\delta^2$$
$$+ 2u_{12}\alpha\beta + 2u_{13}\alpha\gamma + 2u_{14}\alpha\delta + 2u_{23}\beta\gamma + 2u_{24}\beta\delta + 2u_{34}\gamma\delta = 0 \;\ldots\text{(i.)};$$

$\alpha, \beta, \gamma, \delta$, the coordinates of a point P, being the ratios of the tetrahedra $PBCD$, $PCDA$, $PDAB$, $PABC$ to the tetrahedron of reference $ABCD$, and consequently $\alpha+\beta+\gamma+\delta = 1$.

Then, $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ being coordinates of the centre (the pole of the plane at infinity $\alpha+\beta+\gamma+\delta = 0$), we have, as usual,

$$\left.\begin{array}{l} u_{11}\bar{\alpha} + u_{12}\bar{\beta} + u_{13}\bar{\gamma} + u_{14}\bar{\delta} + \epsilon = 0 \\ u_{21}\bar{\alpha} + u_{22}\bar{\beta} + u_{23}\bar{\gamma} + u_{24}\bar{\delta} + \epsilon = 0 \\ u_{31}\bar{\alpha} + u_{32}\bar{\beta} + u_{33}\bar{\gamma} + u_{34}\bar{\delta} + \epsilon = 0 \\ u_{41}\bar{\alpha} + u_{42}\bar{\beta} + u_{43}\bar{\gamma} + u_{44}\bar{\delta} + \epsilon = 0 \\ \quad\;\; \bar{\alpha} + \quad\; \bar{\beta} + \quad\; \bar{\gamma} + \quad\; \bar{\delta} \qquad = 1 \end{array}\right\} \;\ldots\ldots\ldots\ldots\ldots\text{(ii.)},$$

where $u_{21} = u_{12}$, &c., and ϵ is some constant.

2. Let H, h_1, h_2, h_3, h_4 be the discriminants of the surface and the four conics (real or imaginary), in which the surface is cut by the four faces of the tetrahedron of reference $ABCD$; also let B, b_1, b_2, b_3, b_4 be the above discriminants (expressed as determinants) bordered with 1's; thus

$$B \equiv \begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & u_{14}, & 1 \\ u_{21}, & u_{22}, & u_{23}, & u_{24}, & 1 \\ u_{31}, & u_{32}, & u_{33}, & u_{34}, & 1 \\ u_{41}, & u_{42}, & u_{43}, & u_{44}, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix}, \quad b_1 = \begin{vmatrix} u_{22}, & u_{23}, & u_{24}, & 1 \\ u_{32}, & u_{33}, & u_{34}, & 1 \\ u_{42}, & u_{43}, & u_{44}, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}, \text{ \&c.},$$

so that $b_1\ldots$ are the minors of $u_{11}\ldots$ in B. Lastly, let k_1 be the minor of the 1 which is situated in the same row or column as u_{11}, $k_2 =$ &c.; that is,

$$k_1 = \begin{vmatrix} u_{21}, & u_{22}, & u_{23}, & u_{24} \\ u_{31}, & u_{32}, & u_{33}, & u_{34} \\ u_{41}, & u_{42}, & u_{43}, & u_{44} \\ 1, & 1, & 1, & 1 \end{vmatrix}.$$

By a known property of determinants we have the following relations:—

$$\left.\begin{array}{l} k_1^2 = Hb_1 - Bh_1 \\ k_2^2 = Hb_2 - Bh_2 \\ k_3^2 = Hb_3 - Bh_3 \\ k_4^2 = Hb_4 - Bh_4 \end{array}\right\} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii.)}$$

Cor.—If $H=0$, h_1, h_2, h_3, h_4 are all of opposite sign to B; and if $B=0$, b_1, b_2, b_3, b_4 have all the same sign as H.

3. Solving equations (ii.),

$$\bar{a}=\frac{k_1}{B},\ \bar{\beta}=\frac{k_2}{B},\ \bar{\gamma}=\frac{k_3}{B},\ \bar{\delta}=\frac{k_4}{B},\ e=\frac{H}{B};$$

and, multiplying equations (ii.) by $\bar{a}, \bar{\beta}, \bar{\gamma}, \bar{\delta}, -e$, respectively, and adding, we get
$$\phi(\bar{a}\bar{\beta}\bar{\gamma}\bar{\delta})=-e=-\frac{H}{B};$$
thus the equation to the asymptotic cone is (since it is satisfied by the coordinates of the centre of the surface)

$$\begin{aligned}\phi(a\beta\gamma\delta) &= \text{constant}\\ &= \phi(\bar{a}\bar{\beta}\bar{\gamma}\bar{\delta})\\ &= -\frac{H}{B}(a+\beta+\gamma+\delta)^2,\end{aligned}$$

or
$$B\phi(a\beta\gamma\delta)+H(a+\beta+\gamma+\delta)^2=0 \quad\text{..............(iv.)};$$
but, if equation (i.) represent a cone, $H=0$, and the equation becomes
$$\phi(a,\beta,\gamma,\delta)=0,$$
i.e., the asymptotic cone is the surface itself.

4. Through the centre of the surface draw a plane $B'C'D'$ parallel to BCD, and let the coordinates of a point P be $a', \beta', \gamma', \delta'$, referred to the tetrahedron $AB'C'D'$, and a, β, γ, δ referred to $ABCD$.

Let p_1, p_2, p_3, p_4 be the perpendiculars from A, B, C, D on the opposite faces of the tetrahedron $ABCD$, and p_1', p_2', p_3', p_4' the perpendiculars from A, B', C', D' on the opposite faces of the tetrahedron $AB'C'D'$; then ap_1, $a'p_1'$ are the perpendiculars from P on the planes BCD and $B'C'D'$, so that

$$ap_1-a'p_1'=p_1-p_1',$$

or
$$(a-1)p_1=(a'-1)p_1';$$

but when $a=\bar{a}$,
$$a'=0,$$

so that
$$(\bar{a}-1)p_1=-p_1';$$

whence
$$\begin{aligned}a &= a'+\bar{a}(1-a')\\ &= a'+\bar{a}(\beta'+\gamma'+\delta').\end{aligned}$$

Also
$$\beta p_2=\beta' p_2',$$

so that
$$\beta=\frac{p_2'}{p_2}\beta'=\frac{p_1'}{p_1}\beta'=(1-\bar{a})\beta';$$

similarly $$\gamma = (1-\bar{a})\,\gamma',$$

and $$\delta = (1-\bar{a})\,\delta'.$$

Thus, suppressing the accents, we may write the equation to the surface ($AB'C'D'$ being the tetrahedron of reference),

$$\begin{aligned} u_{11}\{a+\bar{a}(\beta+\gamma+\delta)\}^2 &+ \{u_{22}\beta^2+u_{33}\gamma^2+u_{44}\delta^2\}(1-\bar{a})^2 \\ &+2\{a+\bar{a}(\beta+\gamma+\delta)\}(1-\bar{a})\{u_{12}\beta+u_{13}\gamma+u_{14}\delta\} \\ &+2\{u_{23}\beta\gamma+u_{24}\beta\delta+u_{34}\gamma\delta\}(1-\bar{a})^2 = 0. \end{aligned}$$

5. Making now $a = 0$, the above becomes the equation (in triangular coordinates) to the conic F_1 made by the central section of the surface parallel to the face BCD of the original tetrahedron. This equation is

$$\begin{aligned} \lambda^2 u_{11}(\beta+\gamma+\delta)^2+2\lambda(\beta+\gamma+\delta)(u_{12}\beta+u_{13}\gamma+u_{14}\delta)+u_{22}\beta^2+u_{33}\gamma^2+u_{44}\delta^2 \\ +2\{u_{23}\beta\gamma+u_{24}\beta\delta+u_{34}\gamma\delta\} = 0, \end{aligned}$$

where λ is put for $$\frac{\bar{a}}{1-\bar{a}}.$$

The discriminant of this conic is

$$\begin{vmatrix} \lambda^2u_{11}+2\lambda u_{12}+u_{22}, & \lambda^2u_{11}+\lambda(u_{12}+u_{13})+u_{23}, & \lambda^2u_{11}+\lambda(u_{12}+u_{14})+u_{24} \\ \lambda^2u_{11}+\lambda(u_{12}+u_{13})+u_{23}, & \lambda^2u_{11}+2\lambda u_{13}+u_{33}, & \lambda^2u_{11}+\lambda(u_{13}+u_{14})+u_{34} \\ \lambda^2u_{11}+\lambda(u_{12}+u_{14})+u_{24}, & \lambda^2u_{11}+\lambda(u_{13}+u_{14})+u_{34}, & \lambda^2u_{11}+2\lambda u_{14}+u_{44} \end{vmatrix},$$

which is easily seen to be the same as

$$-\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & u_{14}, & -1 \\ u_{21}, & u_{22}, & u_{23}, & u_{24}, & \lambda \\ u_{31}, & u_{32}, & u_{33}, & u_{34}, & \lambda \\ u_{41}, & u_{42}, & u_{43}, & u_{44}, & \lambda \\ -1, & \lambda, & \lambda, & \lambda, & 0 \end{vmatrix}$$

$$= -\lambda^2\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & u_{14}, & 0 \\ u_{21}, & u_{22}, & u_{23}, & u_{24}, & 1 \\ u_{31}, & u_{32}, & u_{33}, & u_{34}, & 1 \\ u_{41}, & u_{42}, & u_{43}, & u_{44}, & 1 \\ 0, & 1, & 1, & 1, & 0 \end{vmatrix} +2\lambda\begin{vmatrix} u_{21}, & u_{22}, & u_{23}, & u_{24} \\ u_{31}, & u_{32}, & u_{33}, & u_{34} \\ u_{41}, & u_{42}, & u_{43}, & u_{44} \\ 0, & 1, & 1, & 1 \end{vmatrix} + \begin{vmatrix} u_{22}, & u_{23}, & u_{24} \\ u_{32}, & u_{33}, & u_{34} \\ u_{42}, & u_{43}, & u_{44} \end{vmatrix}$$

$$= -\lambda^2\{B-2k_1-h_1\}+2\lambda\{k_1+h_1\}+h_1$$
$$= h_1\{1+\lambda\}^2+2\lambda(1+\lambda)\,k_1-B\lambda^2.$$

But since $$\lambda = \frac{\bar{a}}{1-\bar{a}} = \frac{k_1}{B-k_1},$$

$$\frac{1+\lambda}{\lambda} = \frac{B}{k_1};$$

hence the discriminant $= \lambda^2 \left\{ h_1 \frac{B^2}{k_1^2} + 2B - B \right\}$

$$= \lambda^2 B \cdot \frac{Bh_1 + k_1^2}{k_1^2}$$

$$= \lambda^2 B \cdot \frac{Hb_1}{k_1^2} \text{ by (iii.)}$$

$$= \frac{BHb_1}{(B-k_1)^2} \quad \text{..................(v.)}$$

The bordered discriminant of the same conic

$$= \begin{vmatrix} \lambda^2 u_{11} + 2\lambda u_{12} + u_{22}, & \lambda^2 u_{11} + \lambda(u_{12}+u_{13}) + u_{23}, & \lambda^2 u_{11} + \lambda(u_{12}+u_{14}) + u_{24}, & 1 \\ \lambda^2 u_{11} + \lambda(u_{12}+u_{13}) + u_{23}, & \lambda^2 u_{11} + 2\lambda u_{13} + u_{33}, & \lambda^2 u_{11} + \lambda(u_{13}+u_{14}) + u_{34}, & 1 \\ \lambda^2 u_{11} + \lambda(u_{12}+u_{14}) + u_{24}, & \lambda^2 u_{11} + \lambda(u_{13}+u_{14}) + u_{34}, & \lambda^2 u_{11} + 2\lambda u_{14} + u_{44}, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}$$

$$= \begin{vmatrix} u_{22}, & u_{23}, & u_{24}, & 1 \\ u_{32}, & u_{33}, & u_{34}, & 1 \\ u_{42}, & u_{43}, & u_{44}, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = b_1 \quad \text{..................(vi.)};$$

whence we obtain for the area of this conic the expression

$$2\pi\Delta_1' \frac{BHb_1}{\{B-k_1\}^2} \cdot \frac{1}{\{-b_1\}^{\frac{3}{2}}},$$

where $\Delta_1' =$ area of the triangle $B'C'D'$. If $\Delta_1 =$ area of BCD, then

$$\Delta_1' = \Delta_1 \frac{p_1'^2}{p_1^2} = \Delta_1 (1-\bar{a})^2$$

$$= \Delta_1 \left\{ \frac{B-k_1}{B} \right\}^2;$$

so that the area of the conic

$$= 2\pi\Delta_1 \cdot \frac{Hb_1}{B} \cdot \frac{1}{\{-b_1\}^{\frac{3}{2}}}$$

$$= \frac{2\pi\Delta_1 H}{B\sqrt{-b_1}} \quad \text{..................(vii.)}$$

6. Next, let p be the perpendicular from the centre of the surface on the tangent plane parallel to the plane of the above conic, *i.e.*, parallel to $a = 0$.

Let $a_1, \beta_1, \gamma_1, \delta_1$ be the coordinates ($ABCD$ being the tetrahedron of reference) of the point at which this tangent plane is drawn. Then the equation of the plane through $a_1, \beta_1, \gamma_1, \delta_1$, parallel to $a = 0$, is

$$a = a_1(a+\beta+\gamma+\delta),$$

or $\mu a_1 + \beta + \gamma + \delta = 0$, where $\mu = 1 - \frac{1}{a_1}$, or $a_1 = \frac{1}{1-\mu}$,

and
$$p = p_1(\alpha_1 - \bar{\alpha}),$$
$$= p_1\left\{\frac{1}{1-\mu} - \frac{k_1}{B}\right\}$$
$$= p_1\left\{\frac{B-(1-\mu)k_1}{B(1-\mu)}\right\}$$

But the condition that $\mu\alpha+\beta+\gamma+\delta = 0$ should touch the quadric $\phi(\alpha, \beta, \gamma, \delta) = 0$ is

$$\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & u_{14}, & \mu \\ u_{21}, & u_{22}, & u_{23}, & u_{24}, & 1 \\ u_{31}, & u_{32}, & u_{33}, & u_{34}, & 1 \\ u_{41}, & u_{42}, & u_{43}, & u_{44}, & 1 \\ \mu, & 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

which becomes, as we have seen (§ 5),

$$0 = h_1\left\{1-\frac{1}{\mu}\right\}^2 - 2\frac{1}{\mu}\left\{1-\frac{1}{\mu}\right\}k_1 - B\frac{1}{\mu^2};$$

whence
$$B-2k_1(1-\mu) = h_1(1-\mu)^2,$$
or
$$\{B-k_1(1-\mu)\}^2 = (Bh_1-k_1^2)(1-\mu)^2$$
$$= Hb_1(1-\mu)^2 \quad \text{by (iii.)};$$
hence
$$p = \pm\frac{p_1}{B}\sqrt{Hb_1} \quad \text{.......................... (viii.)}$$

And since the area of the parallel central section has been shown to be

$$= \frac{2\pi\Delta_1 H}{B\sqrt{-b_1}},$$

the volume of the quadric (if an ellipsoid)

$$= \frac{4}{3}\cdot\frac{p_1}{B}\sqrt{Hb_1}\cdot\frac{2\pi\Delta_1 H}{\sqrt{-b_1}},$$
$$= 8\pi V\frac{\{-H\}^{\frac{3}{2}}}{B^2},$$

where V = volume of the tetrahedron of reference.

7. I have remarked (*Messenger of Mathematics*, Vol. x., Oct. 1880), that
$$ux^2+vy^2+wz^2+2u'yz+2v'zx+2w'xy = 0$$
being the equation (in triangular coordinates) of a conic, if the three expressions
$$v+w-2u', \; w+u-2v', \; u+v-2w'$$
have not all the same sign, the conic is a hyperbola; if they have the same sign, and the bordered discriminant be negative, the conic is imaginary or real, according as the common sign is or is not the same as that of the discriminant.

The three corresponding expressions for the conic F_1 (§ 5) are
$$\{\lambda^2u_{11}+2\lambda u_{13}+u_{33}\}+\{\lambda^2u_{11}+2\lambda u_{14}+u_{44}\}-2\{\lambda^2u_{11}+\lambda(u_{13}+u_{14})+u_{34}\}, \text{\&c.},$$

or $$u_{33}+u_{44}-2u_{34},\ u_{44}+u_{22}-2u_{42},\ u_{22}+u_{33}-2u_{23}.$$
Hence, if the quantities $u_{11}+u_{22}-2u_{12}\ (\equiv v_{12})$, $u_{11}+u_{33}-2u_{13}\equiv (v_{13})$, &c. are not all of the same sign, one at least of the central sections of the surface parallel to the faces of the tetrahedron of reference is a hyperbola.

8. The area of the conic
$$F_1 = 2\pi\Delta_1 \cdot \frac{H}{B\sqrt{-b_1}} = A_1,$$
the perpendicular from the centre of the surface on the parallel tangent plane
$$= \pm \frac{p_1}{B}\sqrt{Hb_1} = \pi_1,$$
and the discriminant
$$= \frac{BHb_1}{(B-k_1)^2} = D_1.$$
Let the quadric $\phi = 0$ be—

(i.) *An ellipsoid.*

A_1, π_1 are both real, b_1 is negative, and v_{34}, v_{42}, v_{23} are all of the same sign, and unlike that of the discriminant D_1.
Hence H is negative, and v_{34}, v_{42}, v_{23} are of opposite sign to B.

(ii.) *A hyperboloid of one sheet.*

(α.) F_1 an ellipse, A_1 is real, π_1 imaginary, and b_1 negative.
Hence H is positive, and v_{34}, v_{42}, v_{23} have sign opposite to D_1, and therefore the same as B.

(β.) F_1 a hyperbola. A_1 is imaginary, π_1 real, and b_1 positive.
Hence H is positive.

(iii.) *A hyperboloid of two sheets.*

(α.) F_1 a hyperbola. A_1 is imaginary, π_1 real, and b_1 positive.
Hence H is negative.

(β.) F_1 imaginary. A_1 is real, π_1 real, and b_1 negative.
Hence H is negative, and v_{34}, v_{42}, v_{23} have the same sign as D_1, and therefore as B.

(iv.) *Imaginary.*

F_1 is imaginary, A_1 real, π_1 imaginary, and b_1 negative.
Hence H is positive, and v_{34}, v_{42}, v_{23} have the same sign as D_1, and therefore opposite to B.

(v.) *An elliptic paraboloid.*

The surface may be considered as an ellipsoid or hyperboloid of two sheets, having its centre at infinity.
Hence $B=0$ and H is negative; this, by (iii.) § 2, necessitates b_1, b_2, b_3, b_4 all being negative.

(vi.) *A hyperbolic paraboloid.*

This is the limiting case of a hyperboloid of one sheet with its centre at infinity.
Hence $B=0$ and H is positive; in this case b_1, b_2, b_3, b_4 are all positive.

(vii.) *A cone.*

$H = 0$, by § 3. Also h_1, h_2, h_3, h_4 are of opposite sign to B, and one at least not zero, since $\sqrt{-\frac{h_1}{B}}, \sqrt{-\frac{h_2}{B}}, \sqrt{-\frac{h_3}{B}}, \sqrt{-\frac{h_4}{B}}$ are the coordinates of the vertex.

(viii.) *A cylinder.*

$H = 0$, $B = 0$, since the surface has a line of centres. The cylinder will be

$$\left.\begin{array}{llll} \text{Parabolic, if } b_1, \text{ \&c. each} = 0 \\ \text{Elliptic} \quad ,, \quad ,, \quad \text{be negative} \\ \text{Hyperbolic} \;,, \quad ,, \quad \text{be positive} \end{array}\right\} ; \text{ but}$$

(ix.) *Two planes*, if in addition to

$H = 0$, $B = 0$, we have h_1, h_2, h_3, h_4 all $= 0$.

9. If, then, we wish to examine the nature of the surface

$$\phi(\alpha, \beta, \gamma, \delta) = 0,$$

we must first find the values of the quantities

$$B, b_1, \ldots, H, h_1 \ldots v_{12}, \ldots,$$

as defined in § 2 and 7.

Suppose (1.) $B = 0$, $H = 0$.

If h_1, &c. each $= 0$, the surface consists of two planes, which will be parallel if $b_1 \ldots$ are also zero.

If h_1, &c. be not all $= 0$, the surface is a parabolic, elliptic, or hyperbolic cylinder, according as $b_1 \ldots$ are zero, negative, or positive.

(2.) $B \neq 0$, $H = 0$.

The surface is a cone.

(3.) $B = 0$, $H \neq 0$.

The surface is an elliptic or hyperbolic paraboloid, according as H is negative or positive.

(4.) $B \neq 0$, $H \neq 0$.

The surface is central.

(i.) If $v_{12} \ldots$ have not all the same sign, or if any of the quantities b_1, b_2, b_3, b_4 be positive, the surface has hyperbolic sections, and is a hyperboloid of one or two sheets according as H is positive or negative.

(ii.) If $v_{12} \ldots$ have all the same sign, and $b_1 \ldots$ be all negative, then, if H be positive and Bv_{12}, &c. be positive, the surface is a hyperboloid of one sheet; if H be positive and Bv_{12}, &c. be negative, the surface is imaginary; if H be negative and Bv_{12}, &c. be positive, the surface is a hyperboloid of two sheets; if H be negative and Bv_{12}, &c. be negative, the surface is an ellipsoid.

The condition that the surface may be a sphere is given in the ordinary text books.

Thursday, May 12th, 1881.

S. ROBERTS, Esq., F.R.S., President, in the Chair.

The following were elected members:—Lionel Rosenthal, B.A., T.C.D.; Fabian Franklin, Ph.D., Johns Hopkins University; Charles A. van Velzer, Johns Hopkins University; and Miss Christine Ladd, Johns Hopkins University.

Prof. Charles Niven, F.R.S., was admitted into the Society.

The following communications were made:—

"On Ptolemy's Theorem," Mr. C. W. Merrifield.

"The Summation of certain Hyper-geometric Series," Rev. T. R. Terry.

"Quaternion Proof of Mr. S. Roberts's Theorem of Four Co-intersecting Spheres," Mr. J. J. Walker.

"Some Solutions of the '15-girl' Problem," Mr. E. Carpmael.

"Note on the Coordinates of a Tangent Line to the Curve of Intersection of two Quadrics," Mr. W. R. W. Roberts.

Short communications were made by the President, and Messrs. Cayley, Hart, and Walker.

The following presents were made to the Library:—

"Educational Times," May, 1881.

"Notice sur les travaux géométriques de M. Mannhein (à l'appui de sa candidature à l'Académie des Sciences)," from the Author; Paris, 1881.

"List of Members of the Institution of Civil Engineers," 30th March, 1881.

"Hyper-elliptische Functionen und die Kummer'sche Fläche," von K. Rohn (Math. Ann., Band. xv., pp. 1–40).

"Die verschiedenen Gestalten der Kummer'schen Fläche," von Karl Rohn (Math. Ann., Band xviii., pp. 99–159).

"Théorie des figures projectives sur une surface du second ordre," par H. G. Zeuthen, à Copenhague (Math. Ann., Band xviii., pp. 33–68).

"Konstruktion af det ottende Skærings Punkt, mellem de Fladen af anden Orden, som gaa gjennem syv givne Punkter," af H. G. Zeuthen.

"Construction du huitième point commun aux surfaces du second ordre qui passent par sept points donnés," par H. G. Zeuthen.

"Théorie des figures projectives sur une surface du second ordre," par H. G Zeuthen.

"Grafisk Behandlung af en Bjælkes bevægelige Belastning," af H. G. Zeuthen: Kjøbenhavn, 1881.

"Atti della R. Accademia dei Lincei—Transunti," Vol. v., Fasc. 9, 10; Roma, 1881.

"Monatsbericht," Dec., 1880.

"Proceedings of Royal Society," Vol. xxxi., No. 211.

"Beiblätter zu den Annalen der Physik und Chemie," Band v., Stück 4; Leipzig, 1881.

"Crelle," Band 91, 1es Heft; Berlin, 1881.

"Bulletin des Sciences Mathématiques et Astronomiques," Tome iv., Nov., Dec., 1880.

"Proceedings of the Cambridge Philosophical Society," Vol. iii., Pt. vii., Oct. to Dec., 1879; Pt. viii., Feb. to May, 1880.

"Transactions of the Cambridge Philosophical Society," Vol. xiii., Pt. i., with Index to Vols. i. to xii.

"Proceedings of the Royal Society of Edinburgh," Session 1879–1880, Vol. x., pp. 315–781.

Note on the Coordinates of a Tangent Line to the Curve of Intersection of two Quadrics. By W. R. Westropp Roberts.

[*Read May 12th,* 1881.]

The condition that the line, the six coordinates of which are a, b, c, f, g, h, should touch the surface

$$U \equiv \alpha x^2+\beta y^2+\gamma z^2+\delta u^2,$$

is $$1^\circ. \ldots\ldots \ \delta\,(\alpha f^2+\beta g^2+\gamma h^2)+\beta\gamma a^2+\gamma\alpha b^2+\alpha\beta c^2=0.$$

I now determine the conditions that the line (a, b, c, f, g, h) should be a generator of U. Solving for x and y from the equations of the line, and substituting the resulting values in the equation of the quadric, we get a quadratic in z and u, each coefficient of which we must equate to zero. We have then three conditions which, joined to the universal equation $af+bg+ch=0$, give us

$$\left\{\begin{array}{l} 2^\circ. \ldots\ldots \ \dfrac{a^2}{f^2}=\dfrac{\alpha\delta}{\beta\gamma},\quad \dfrac{b^2}{g^2}=\dfrac{\beta\delta}{\alpha\gamma},\quad \dfrac{c^2}{h^2}=\dfrac{\gamma\delta}{\alpha\beta}, \\ 3^\circ. \ldots\ldots \ \alpha f^2+\beta g^2+\gamma h^2=0. \end{array}\right.$$

Now, if $V\equiv\alpha' x^2+\beta' y^2+\gamma' z^2+\delta' u^2$; through any assumed point we can describe a quadric of the form $U+\lambda V$, the two generators of which, passing through the assumed point, are "lines through two points" of the curve UV.

Hence the line (a, b, c, f, g, h) will pass through two points of the

curve UV if the following conditions are fulfilled:—

$$4^\circ \ldots\ldots \left\{ \begin{array}{l} \dfrac{a^2}{f^2} = \dfrac{(\delta+\lambda\delta')(\alpha+\lambda\alpha')}{(\beta+\lambda\beta')(\gamma+\lambda\gamma')}, \quad \dfrac{b^2}{g^2} = \dfrac{(\delta+\lambda\delta')(\beta+\lambda\beta')}{(\gamma+\lambda\gamma')(\alpha+\lambda\alpha')}, \\ \qquad\qquad \dfrac{c^2}{h^2} = \dfrac{(\delta+\lambda\delta')(\gamma+\lambda\gamma')}{(\alpha+\lambda\alpha')(\beta+\lambda\beta')}, \end{array} \right.$$

$$5^\circ.\ (\alpha+\lambda\alpha')f^2+(\beta+\lambda\beta')g^2+(\gamma+\lambda\gamma')h^2 = 0.$$

But if the line (a, b, c, f, g, h) be a tangent to the curve UV, it will touch both U and V, so that we obtain the additional conditions, by (1°), viz.,

$$6^\circ \ldots\ldots \left\{ \begin{array}{l} \delta(\alpha f^2+\beta g^2+\gamma h^2)+\beta\gamma a^2+\gamma\alpha b^2+\alpha\beta c^2 = 0, \\ \delta'(\alpha' f^2+\beta' g^2+\gamma' h^2)+\beta'\gamma' a^2+\gamma'\alpha' b^2+\alpha'\beta' c^2 = 0. \end{array} \right.$$

From these equations we deduce easily the following values for the coordinates of a tangent line to the curve UV, in terms of one parameter λ,

$$7^\circ \ldots\ldots\ldots\ldots \left\{ \begin{array}{l} a = \{(\beta\gamma')(\alpha\delta')(\alpha+\lambda\alpha')(\delta+\lambda\delta')\}^{\frac{1}{2}}, \\ b = \{(\gamma\alpha')(\beta\delta')(\beta+\lambda\beta')(\delta+\lambda\delta')\}^{\frac{1}{2}}, \\ c = \{(\alpha\beta')(\gamma\delta')(\gamma+\lambda\gamma')(\delta+\lambda\delta')\}^{\frac{1}{2}}, \\ f = \{(\beta\gamma')(\alpha\delta')(\beta+\lambda\beta')(\gamma+\lambda\gamma')\}^{\frac{1}{2}}, \\ g = \{(\gamma\alpha')(\beta\delta')(\gamma+\lambda\gamma')(\alpha+\lambda\alpha')\}^{\frac{1}{2}}, \\ h = \{(\alpha\beta')(\gamma\delta')(\alpha+\lambda\alpha')(\beta+\lambda\beta')\}^{\frac{1}{2}}, \end{array} \right.$$

where $(\beta\gamma') = \beta\gamma'-\gamma\beta'$, $(\alpha\delta') = \alpha\delta'-\delta\alpha'$, &c.;

or, adopting Cayley's notation,

$$\begin{array}{l} a = \{AF(\alpha+\lambda\alpha')(\delta+\lambda\delta')\}^{\frac{1}{2}}, \\ b = \{BG(\beta+\lambda\beta')(\delta+\lambda\delta')\}^{\frac{1}{2}}, \\ c = \{CF(\gamma+\lambda\gamma')(\delta+\lambda\delta')\}^{\frac{1}{2}}, \\ f = \{AF(\beta+\lambda\beta')(\gamma+\lambda\gamma')\}^{\frac{1}{2}}, \\ g = \{BG(\gamma+\lambda\gamma')(\alpha+\lambda\alpha')\}^{\frac{1}{2}}, \\ h = \{CF(\alpha+\lambda\alpha')(\beta+\lambda\beta')\}^{\frac{1}{2}}. \end{array}$$

We can now find the equation of the developable generated by the tangent lines to the curve UV, by substituting the above values of the coordinates in any one of the four equations of the line, λ being determined by the equation $U+\lambda V = 0$.

From the equation $ax+by+cz = 0$, we find, for one form of the equation of the developable,

$$8^\circ \ldots\ldots\ldots\ldots x\{AF(\alpha V-\alpha' U)\}^{\frac{1}{2}}+y\{BG(\beta V-\beta' U)\}^{\frac{1}{2}} +z\{CF(\gamma V-\gamma' U)\}^{\frac{1}{2}} = 0.$$

The sections, then, by the principal planes, are at once shown to be double lines, which are trinodal quartics.

Moreover, the above form shows that UV is a double line on the developable, and that the locus meets U again in eight right lines.

Forming the equation of the plane containing two consecutive lines of the system, I find the equation of the osculating plane to be

$$9^\circ \ldots\ldots \left\{ \begin{aligned} & x(a+\lambda a')^{\frac{3}{2}}(AGH)^{\frac{1}{2}} + y(\beta+\lambda\beta')^{\frac{3}{2}}(BFH)^{\frac{1}{2}} \\ & + z(\gamma+\lambda\gamma')^{\frac{3}{2}}(CFG)^{\frac{1}{2}} + w(\delta+\lambda\delta')^{\frac{3}{2}}(ABC)^{\frac{1}{2}} = 0. \end{aligned} \right.$$

We can, without difficulty, verify that the cuspidal edge of the developable generated by the plane is the curve UV.

It is easy to see, also, that the coordinates of any point on UV can be expressed as follows:

$$10^\circ \ldots\ldots\ldots\ldots \left\{ \begin{aligned} \theta x &= \{(a+\lambda a')\,AGH\}^{\frac{1}{2}}, \\ \theta y &= \{(\beta+\lambda\beta')\,BFH\}^{\frac{1}{2}}, \\ \theta z &= \{(\gamma+\lambda\gamma')\,CFG\}^{\frac{1}{2}}, \\ \theta w &= \{(\delta+\lambda\delta')\,ABC\}^{\frac{1}{2}}. \end{aligned} \right.$$

If, then, $v = \displaystyle\int_0^\lambda \frac{\theta\lambda}{\sqrt{(a+\lambda a')(\beta+\lambda\beta')(\gamma+\lambda\gamma')(\delta+\lambda\delta')}}$,

any plane meets the curve UV in four points, such that

$$11^\circ \ldots\ldots\ldots\ldots v_1+v_2+v_3+v_4 = \text{constant} = c.$$

By putting $\qquad v_1 = v_2 = v_3 = v_4 = v,$

we find $\qquad 4v = c,$

which gives the 16 stationary tangent planes. By means of the equation (11°), we can find many theorems analogous to those in plane cubics which depend on the fact that a curve of the m^{th} degree meets it in $3m$ points, the sum of the amplitudes of which is constant.

It is to be observed that the preceding formulæ admit of a two-fold interpretation, and that when we know the coordinates of a point, a line, and an osculating plane of the curve UV, in terms of a parameter λ, we know the coordinates of a tangent plane, a line, and a point on the cuspidal edge of the common tangent developable of U and V.

Hence the coordinates of a point on the cuspidal edge of the developable circumscribing U and V can be written

$$12^\circ \ldots\ldots\ldots\ldots \left\{ \begin{aligned} x &= \theta\,(a+\lambda a')^{\frac{3}{2}}(AGH)^{\frac{1}{2}}, \\ y &= \theta\,(\beta+\lambda\beta')^{\frac{3}{2}}(BFH)^{\frac{1}{2}}, \\ z &= \theta\,(\gamma+\lambda\gamma')^{\frac{3}{2}}(CFG)^{\frac{1}{2}}, \\ w &= \theta\,(\delta+\lambda\delta')^{\frac{3}{2}}(ABC)^{\frac{1}{2}}. \end{aligned} \right.$$

Eliminating λ from the first three of the above equations, we find

$$13^\circ \ldots\ldots\ldots\ldots x^{\frac{2}{3}}A^{\frac{2}{3}}E^{\frac{1}{3}} + y^{\frac{2}{3}}B^{\frac{2}{3}}G^{\frac{1}{3}} + z^{\frac{2}{3}}C^{\frac{2}{3}}E^{\frac{1}{3}} = 0.$$

Or, the cuspidal edge of the common tangent developable is projected on any one of the principal planes into the quasi-evolute of a conic.

To investigate the conic of which equation (13) is the quasi-evolute, consider Cayley's form of the quasi-normal, viz.,

$$14^\circ.\quad (Lx_1+My_1+Nz_1)\begin{vmatrix} x & y & z \\ x' & y' & z' \\ x_2 & y_2 & z_2 \end{vmatrix} + (Lx_2+My_2+Nz_2)\begin{vmatrix} x & y & z \\ x' & y' & z' \\ x_1 & y_1 & z_1 \end{vmatrix} = 0,$$

$x_1y_1z_1$, $x_2y_2z_2$ being the coordinates of the IJ points.

Now, from the above equation, the quasi-normal of $\mathrm{a}x^2+\mathrm{b}y^2+\mathrm{c}z^2=0$ is found to be

$$15^\circ.\quad (\mathrm{b}y_1y_2-\mathrm{c}z_1z_2)\frac{x}{x'}+(\mathrm{c}z_1z_2-\mathrm{a}x_1x_2)\frac{y}{y'}+(\mathrm{a}x_1x_2-\mathrm{b}y_1y_2)\frac{z}{z'}=0,$$

$$\text{if } 16^\circ \ldots\ldots\ldots\ldots \begin{cases} x_1y_2+y_1x_2=0, \\ y_1z_2+z_1y_2=0, \\ z_1x_2+z_2x_1=0, \end{cases}$$

and the envelope of (15°) is

$$17^\circ.\quad x^{\frac{2}{3}}(\mathrm{b}y_1y_2-\mathrm{c}z_1z_2)^{\frac{2}{3}}\,\mathrm{a}^{\frac{1}{3}}+y^{\frac{2}{3}}(\mathrm{c}z_1z_2-\mathrm{a}x_1x_2)^{\frac{2}{3}}\,\mathrm{b}^{\frac{1}{3}}+z^{\frac{2}{3}}(\mathrm{a}x_1x_2-\mathrm{b}y_1y_2)^{\frac{2}{3}}\,\mathrm{c}^{\frac{1}{3}}=0,$$

and, by comparison of coefficients in (13°) and (17°), we find

$$18^\circ \ldots\ldots\ldots\ldots\ \mathrm{a}=\frac{1}{F},\quad \mathrm{b}=\frac{1}{G},\quad \mathrm{c}=\frac{1}{H},$$

$$19^\circ.\quad \frac{y_1y_2}{G}-\frac{z_1z_2}{H}=AF,\quad \frac{z_1z_2}{H}-\frac{x_1x_2}{F}=BG,\quad \frac{x_1x_2}{F}-\frac{y_1y_2}{G}=CH.$$

The system (16°) shows that the IJ points must lie on an edge of the self-conjugate tetrahedron of reference, so that the projection of the cuspidal edge of the developable circumscribed to U and V on the principal plane $U=0$, is the quasi-evolute of the conic

$$\frac{x^2}{F}+\frac{y^2}{G}+\frac{z^2}{H}=0,$$

the IJ points being any one of the three pairs of points given by the equations (16°) and (19°).

The coordinates of a line of the system will be found to be

$$20^\circ \ldots\ldots\ldots \begin{cases} a=\{AF(\beta+\lambda\beta')(\gamma+\lambda\gamma')\}^{\frac{1}{2}}, \\ b=\{BC(\gamma+\lambda\gamma')(\alpha+\lambda\alpha')\}^{\frac{1}{2}}, \\ c=\{CH(\alpha+\lambda\alpha')(\beta+\lambda\beta')\}^{\frac{1}{2}}, \\ f=\{AF(\alpha+\lambda\alpha')(\delta+\lambda\delta')\}^{\frac{1}{2}}, \\ g=\{BG(\beta+\lambda\beta')(\delta+\lambda\delta')\}^{\frac{1}{2}}, \\ h=\{CH(\gamma+\lambda\gamma')(\delta+\lambda\delta')\}^{\frac{1}{2}}. \end{cases}$$

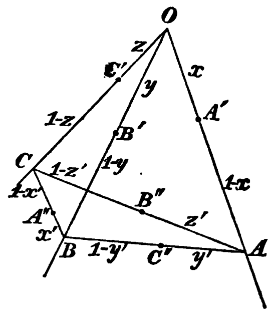
The above values are deduced at once from (7°), for if the coordinates of a line be a, b, c, f, g, h, the coordinates of the reciprocal line with regard to the surface $x^2+y^2+v^2+u^2=0$, will be

$$f, g, h, a, b, c.$$

The coordinates of a tangent plane follow at once from (10°).

Quaternion Proof of Mr. Samuel Roberts' Theorem of Four Co-intersecting Spheres. By J. J. Walker, M.A.

[*Read May* 12*th*, 1881.]



If, taking O as origin, three vectors OA (α), OB (β), OC (γ) determine a tetrahedron, and if on these three arbitrary points A', B', C' are taken, the four points O, A', B', C' determine a sphere (S_1). If on BC, CA, AB, also, arbitrary points A'', B'', C'' are taken, then A, A', B'', C'' determine a second sphere (S_2); B, A'', B', C'' a third sphere (S_3); and C, A'', B'', C' a fourth sphere (S_4). The theorem to be proved is, that these four spheres have a common point of intersection.

Let $\overline{OA'}$, $\overline{OB'}$, $\overline{OC'}$ be $x\alpha$, $y\beta$, $z\gamma$ respectively; then the vector of any point on (S_1) is $\rho = lx\alpha + my\beta + nz\gamma$, where l, m, n satisfy

$$l(l-1)\,x^2\alpha^2 + \ldots + 2mnyzS\beta\gamma + \ldots = 0 \quad \ldots\ldots\ldots\ldots\ldots(1).$$

Similarly the vectors to any points on (S_2), (S_3) respectively are (if $\overline{AB''} = z'\overline{AC}$, $\overline{AC''} = y'\overline{AB}$, $\overline{BA''} = x'\overline{BC}$),

$$\alpha - \rho = l'(1-x)\,\alpha + m'y'\,(\alpha-\beta) + n'z'\,(\alpha-\gamma),$$

or
$$\rho = (l'x - l' + 1 - m'y' - n'z')\,\alpha + m'y'\beta + n'z'\gamma,$$

with
$$l'(l'-1)\,(1-x)^2\,\alpha^2 + m'(m'-1)\,y'^2\,(\alpha-\beta)^2 + \ldots$$
$$+ 2m'n'S\,(\alpha-\beta)\,(\alpha-\gamma) + \ldots = 0 \quad \ldots\ldots(2);$$

$$\beta - \rho = l''\,(1-y')\,(\beta-\alpha) + m''\,(1-y)\,\beta + n''x'\,(\beta-\gamma),$$

or
$$\rho = l''\,(1-y')\,\alpha + (l''y' - l'' + m''y - m'' - n''x + 1)\,\beta + n''x'\gamma,$$

with
$$l''(l''-1)\,(1-y')^2\,(\beta-\alpha)^2 + m''\,(m''-1)\,(1-y)^2\,\beta^2$$
$$+ n''\,(n''-1)\,x'^2\,(\beta-\gamma)^2 + 2m''n''x'\,(1-y)\,S\beta\,(\beta-\gamma)$$
$$+ 2n''l''x'\,(1-y')\,S\,(\beta-\alpha)\,(\beta-\gamma)$$
$$+ 2l''m''\,(1-y')\,(1-y)\,S\beta\,(\beta-\alpha) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Identifying the ρ of S_1 with that of S_2, $l' = \dfrac{lx + my + nz - 1}{x - 1}$, $m' = \dfrac{my}{y'}$, $n' = \dfrac{nz}{z'}$; and these values, substituted in (2), give

$$[(lx + my + nz - 1)\{(l-1)x + my\,(my - 1 - y') + nz(nz - 1 - z')\} + 2mnyz]\,\alpha^2$$
$$+ my\,(my - y')\,\beta^2 + nz\,(nz - z')\,\gamma^2 + 2mnyzS\beta\gamma$$
$$+ 2\,\{nlzx + nz\,(z'-1)\}\,S\gamma\alpha + 2\,\{lmxy + my\,(y'-1)\}\,S\alpha\beta = 0;$$

or, in virtue of (1),

$$\{x(1-l)+(1-x)(my+nz)-myy'-nzz'\}\alpha^2+my(y-y')\beta^2+nz(z-z')\gamma^2$$
$$+2nz(z'-1)S\gamma\alpha+2my(y'-1)S\alpha\beta = 0 \ldots\ldots\ldots\ldots\ldots(4).$$

Similarly, by identifying the ρ of S_1 with that of S_3, and substituting the values of l'', m'', n'' in (3),

$$lx(x-1+y')\alpha^2+\{y(1-m)-y(nz+lx)+n(1-x')z+ly'x\}\beta^2$$
$$+nz(z-x')\gamma^2+2nz(x'-1)S\beta\gamma-2lxy'S\alpha\beta = 0 \ldots\ldots\ldots(5).$$

Multiplying (4) by lx and (5) by my, and adding, in virtue of (1), there results

$$nz[\{(1-x-z')\alpha^2+(z-z')\gamma^2+2z'S\gamma\alpha\}lx$$
$$+\{(1-x'-y)\beta^2+(z-x')\gamma^2+2x'S\beta\gamma\}my+\gamma^2(n-1)z] = 0\ldots(6).$$

(4), (5), (6) determine two sets of values of l, m, n; in one of which $n=0$; verifying the otherwise known result, that one intersection of $S_1S_2S_3$ lies in the plane OAB. The other set are found from the second factor of (6), together with (4), (5), which may be arranged as

$$\alpha^2(l-1)x+\{(-1+x+y')\alpha^2+(y'-y)\beta^2+2(1-y')S\alpha\beta\}my$$
$$+\{(x+z'-1)\alpha^2+(z'-z)\gamma^2+2(1-z')S\gamma\alpha\}nz = 0 \ldots(7),$$

$$\{(-x-y'+1)\alpha^2+(y-y')\beta^2+2y'S\alpha\beta\}lx+\beta^2(m-1)y$$
$$+\{(x'+y-1)\beta^2+(x'-z)\gamma^2+2(1-x')S\beta\gamma\}nz = 0 \ldots(8).$$

Now the points of intersection of $S_1S_2S_4$ would evidently be determined by changing β into γ, y into z, m into n, y' into z', x' into $(1-x')$, and *vice versâ*, in the above equations; but, by these interchanges, (6) becomes actually $my\,[(8)]$, (7) is unchanged, and (8) becomes the second factor of (6). Thus, that intersection (P) of $S_1S_2S_4$ which does not lie in the plane OCA, is shown to coincide with the intersection of $S_1S_2S_3$, which does not lie in the plane OAB. Similarly, by interchanging α, γ; l, n; x, z; $y', 1-x'$; $z', 1-z'$, we obtain from (6) $lx\,[(7)]$, from (7) the second factor of (6), while (8) remains unchanged. But by these interchanges the triad $S_1S_2S_3$ is changed into $S_1S_3S_4$; and thus S_3S_4 are shown to have one point of intersection with S_1 coinciding with P.

Some Solutions of Kirkman's 15-*school-girl Problem.*

By Ernest Carpmael, M.A.

[*Read May* 12*th*, 1881.]

Several communications with reference to the above problem have, from time to time, appeared in various Mathematical journals, without,

so far as I am aware, any complete solution having been given; the most complete of these being contained in two papers by Mr. W. S. B. Woolhouse, which appeared in the *Lady's and Gentleman's Diary* for 1862 and 1863 respectively. Upon recently referring to the latter, I found several of my results anticipated; but, as some appear to be novel, I send a summary, and trust that the varied yet symmetrical nature of the solutions, and the fact that many eminent mathematicians have dabbled with the problem, may be sufficient excuse for troubling the Society with what is, after all, nothing but an ingenious mathematical puzzle.

The problem may be thus stated,—"Fifteen young ladies in a school walk out three abreast for seven days in succession, it is required to arrange them daily so that no two shall walk twice abreast."

I assume as a standard walk the following (the numbers 1, 2, 3, 4, 5, being prefixed to the several rows in order to characterize each row by a single number):—

1. $a\ b\ c$
2. $d\ e\ f$
3. $g\ h\ i$
4. $j\ k\ l$
5. $m\ n\ p$

It is easily seen that, if we consider one letter, say a, the remaining six pairs of companions of a must be of one or other of the three following forms:—

$$(2\ 3),\ (2\ 4),\ (2\ 5),\ (3\ 4),\ (3\ 5),\ (4\ 5)\ \ldots\ldots\ldots\ldots(\alpha),$$

$$(2\ 3)_2,\ (2\ 4),\ (4\ 5)_2,\ (3\ 5)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\beta),$$

$$(2\ 3)_3,\ (4\ 5)_3\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\gamma),$$

where the notation (2 3), for instance, means that this pair is formed by one element from each of the two rows 2 and 3, as for example $d\,g$, $(2\ 3)_2$ means that two pairs are formed by taking one element from each of the rows 2 and 3, *e.g.*, $d\,g$ and $e\,h$, and it is understood that any of the rows may be interchanged, the one with the other, as this leaves the standard walk unaltered.

It is obvious that, in the most symmetrical solutions which can be formed, all the six pairs of companions of each of the letters must be of the form (α), and in order that this may be the case, three triads must be chosen from each of the ten combinations of the five rows taken three and three together; and it is also further obvious that no real restriction is made, so far as regards these solutions, if we assume a particular set of companions for one latter, say a, because any other set of the same form (α) may be deduced by a simple translation, by which is meant a translation which shall leave the standard walk un-

altered. Thus, assume that a's companions are

$$d\,g,\ e\,j,\ f\,m,\ h\,k,\ i\,n,\ l\,p,$$

and write down in succession all the sets of three triads, which can be formed as above described. We have

	1	2	3	4	5	6	7	8	9	10
A	*a d g* *b e h* *c f i*	*a e j* *b f k* *c d l*	*a f m* *b d n* *c e p*	*a h k* *b i l* *c g j*	*a i n* *b g p* *c h m*	*a l p* *b j m* *c k n*	*d h j* *e i k* *f g l*	*d i m* *e g n* *f h p*	*d k p* *e l m* *f j n*	*g k m* *h l n* *i j p*
B	do.	do.	do.	do.	do.	do.	*d i j* *e g k* *f h l*	*d h p* *e i m* *f g n*	*d k m* *e l n* *f j p*	*g l m* *h j n* *i k p*
C	do.	do.	*a f m* *b d p* *c e n*	*a h k* *b i l* *c g j*	*a i n* *b g m* *c h p*	*a l p* *b j n* *c k m*	*d i k* *e g l* *f h j*	*d h n* *e i m* *f g p*	*d j m* *e k p* *f l n*	*g k n* *h l m* *i j p*
D	do.	*a e j* *b f l* *c d k*	*a f m* *b d n* *c e p*	*a h k* *b i j* *c g l*	*a i n* *b g p* *c h m*	*a l p* *b k n* *c j m*	*d h l* *e i k* *f g j*	*d i p* *e g m* *f h n*	*d j m* *e l n* *f k p*	*g k n* *h j p* *i l m*
E	do.	do.	*a f m* *b d p* *c e n*	*a h k* *b i j* *c g l*	*a i n* *b g m* *c h p*	*a l p* *b k n* *c j m*	*d h j* *e i l* *f g k*	*d i m* *e g p* *f h n*	*d l n* *e k m* *f j p*	*g j n* *h l m* *i k p*
F	do.	do.	do.	do.	do.	do.	*d i l* *e g k* *f h j*	*d h m* *e i p* *f g n*	*d j n* *e l m* *f k p*	*g j p* *h l n* *i k m*
G	*a d g* *b e i* *c f h*	*a e j* *b f k* *c d l*	*a f m* *b d n* *c e p*	*a h k* *b g l* *c i j*	*a i n* *b h p* *c g m*	*a l p* *b j m* *c k n*	*d i k* *e h l* *f g j*	*d h m* *e g n* *f i p*	*d j p* *e k m* *f l n*	*g k p* *h j n* *i l m*
H	do.	do.	*a f m* *b d p* *c e n*	*a h k* *b g l* *c i j*	*a i n* *b h m* *e g p*	*a l p* *b j n* *c k m*	*d h j* *e g k* *f i l*	*d i m* *e h p* *f g n*	*d k n* *e l m* *f j p*	*g j m* *h l n* *i k p*
I	do.	do.	do.	do.	do.	do.	*d i k* *e h l* *f g j*	*d h n* *e g m* *f i p*	*d j m* *e k p* *f l n*	*g k n* *h j p* *i l m*
J	do.	*a e j* *b f l* *c d k*	*a f m* *b d n* *c e p*	*a h k* *b g j* *c i l*	*a i n* *b h p* *c g m*	*a l p* *b k m* *c j n*	*d h j* *e g l* *f i k*	*d i p* *e h m* *f g n*	*d l m* *e k n* *f j p*	*g k p* *h l n* *i j m*
K	do.	do.	do.	do.	do.	do.	*d h l* *e g k* *f i j*	*d i m* *e h n* *f g p*	*d j p* *e l m* *f k n*	*g l n* *h j m* *i k p*
L	do.	do.	*a f m* *b d p* *c e n*	*a h k* *b g j* *c i l*	*a i n* *b h m* *c g p*	*a l p* *b k n* *c j m*	*d i j* *e h l* *f g k*	*d h n* *e g m* *f i p*	*d l m* *e k p* *f j n*	*g l n* *h j p* *i k m*

These 12 sets of ten are all that can be formed consistently with the above conditions. A little examination shows that these are not all

independent, but that some are mere translations of others; thus, if δ represent the operation of translation by means of the following formula:—

$$\delta = \begin{array}{ccc|ccc|ccc|ccc|ccc} a & b & c & d & e & f & g & h & i & j & k & l & m & n & p \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a & b & c & e & f & d & j & l & k & m & p & n & g & h & i \end{array}$$

the following relations will be found to hold, viz.,

$$\left.\begin{array}{l} \delta E = \delta^2 B = K \\ \delta F = \delta^2 A = J \\ \delta C = \delta^2 G = L \end{array}\right\}.$$

It will be observed that the operation δ leaves both the standard walk and a's set of companions unaltered. A little closer examination shows also that A and B give no solution, *i.e.*, cannot be put together so as to form seven walks, and that $C = D = H$, as appears by the application of the following translation formulæ:—

C	$a\ b\ c$	$d\ e\ f$	$g\ h\ i$	$j\ k\ l$	$m\ n\ p$
D	$a\ b\ c$	$g\ h\ i$	$d\ e\ f$	$k\ j\ l$	$n\ m\ p$
H	$p\ m\ n$	$l\ k\ j$	$a\ b\ c$	$i\ g\ h$	$f\ d\ e$

where the letters in the same vertical column correspond.

There remain, therefore, only two possible forms of symmetrical solutions, viz., H and I.

The form I is equivalent to that given in Mr. Woolhouse's first paper, and has often been given before by others. It can be put together in thirty different ways, all consistent with the standard walk remaining unaltered, but these thirty sets of seven walks are not independent, but can be reduced, as was shown by Mr. Woolhouse, to two typical forms, from which all the others can be obtained by simple translation. Thus we may take as the two typical forms,—

	1	δ	δ^2	1	δ	δ^2	
$a\ b\ c$	$a\ d\ g$	$a\ e\ j$	$a\ f\ m$	$a\ h\ k$	$a\ l\ p$	$a\ n\ i$	
$d\ e\ f$	$b\ e\ i$	$b\ f\ k$	$b\ d\ p$	$b\ g\ l$	$b\ j\ n$	$b\ m\ h$	
$g\ h\ i$	$c\ k\ m$	$c\ p\ g$	$c\ i\ j$	$c\ e\ n$	$c\ f\ h$	$c\ d\ l$	......(1),
$j\ k\ l$	$f\ l\ n$	$d\ n\ h$	$e\ h\ l$	$d\ j\ m$	$e\ m\ g$	$f\ g\ j$	
$m\ n\ p$	$h\ j\ p$	$l\ m\ i$	$n\ g\ k$	$f\ i\ p$	$d\ k\ i$	$e\ p\ k$	
				$a\ h\ k$	$a\ l\ p$	$a\ n\ i$	
				$b\ j\ n$	$b\ m\ h$	$b\ g\ l$	
do.	do.	do.	do.	$c\ d\ l$	$c\ e\ n$	$c\ f\ h$	......(2),
				$e\ g\ m$	$f\ j\ g$	$d\ m\ j$	
				$f\ i\ p$	$d\ k\ i$	$e\ p\ k$	

in which it will be observed that the walks in the third and fourth columns, as also those in the sixth and seventh, can be deduced respectively from the walks in the second and fifth columns respectively, by means of the operation δ. It will be observed that the second form differs only in the arrangement of nine triads appearing three together in three walks.

All the other 28 sets of seven walks which can be formed from this particular set of 35 triads, can be deduced from (1) and (2) by the aid of the translation formula δ already given, together with the following additional translation formulæ—

a b c	*d e f*	*g h i*	*j k l*	*m n p*
c b a	*e d f*	*n m p*	*l k j*	*h g i*
a c b	*e d f*	*j k l*	*g h i*	*m p n*
a b c	*j k l*	*e d f*	*h g i*	*p m n*
g i h	*b c a*	*l k j*	*p n m*	*d f e*

or numerous others, suitable for the purpose, can be readily formed.

These two typical forms of I can also be arranged so as to show the curious cyclical property pointed out by Mr. Woolhouse; thus,

c b a	*c d l*	*c e n*	*c p g*	*c f h*	*c k m*	*c i j*
d e f	*e p k*	*p f i*	*f k b*	*k i d*	*i b e*	*b d p*
g h i	*h m b*	*m j d*	*j a e*	*a l p*	*l n f*	*n g k*
j k l	*a i n*	*l b g*	*n d h*	*g e m*	*h p j*	*m f a*
m n p	*j g f*	*a h k*	*l m i*	*n j b*	*g a d*	*h l e*

where the letters in each horizontal row follow in the same order as that given by the table

$$\left.\begin{matrix} b & d & e & p & f & k & i \\ a & l & n & g & h & m & j \end{matrix}\right\} \quad \text{..........................No. 1.}$$

Similarly, the triads can be arranged by means of the table

$$\left.\begin{matrix} c & e & f & n & d & l & h \\ a & i & k & j & p & g & m \end{matrix}\right\} \quad \text{..........................No. 2;}$$

thus,

b c a	*b e i*	*b f k*	*b n j*	*b d p*	*b l g*	*b h m*
d e f	*l f n*	*h n d*	*c d l*	*e l h*	*f h c*	*n c e*
g h i	*m c k*	*a e j*	*i f p*	*k n g*	*j d m*	*p l a*
j k l	*p j h*	*g p e*	*m g e*	*a m f*	*i a n*	*k i d*
m n p	*a d g*	*i l m*	*k h a*	*j c i*	*p e k*	*g f j*

The form H furnishes six complete sets of seven walks; but these again can be reduced to two types, from which the others can be obtained by simple translation.

The following table shows the possible walks, and their mutual relation to one another and to the operation δ :—

1	δ	δ^2
adg bfh cjn elm ikp *bkm cei fjp hln* *cjn ehp fil*	*aej bdl cmh fng kpi* *bpg cfk dmi lnh* *cmh fli dkn*	*afm ben cgl dhj pik* *bij cdp egk nhl* *cgl dnk eph*
ahk ben cdp fil gjm *cgl dim fjp* *bij cdp elm fgn*	*alp bfh cei dkn jmg* *cjn ekg dmi* *bkm cei fng djh*	*ani bdl cfk eph mgj* *cmh fpj egk* *bpg cfk dhj eml*

and the two typical forms of the complete seven walks into which these triads may be arranged, can be written

a b c	*a d g*	*a l p*	*a e j*	*a f m*	*a h k*	*a i n*	
d e f	*b f h*	*b k m*	*b g p*	*b i j*	*b e n*	*b d l*	
g h i	*c j n*	*c e i*	*c f k*	*c g l*	*c d p*	*c h m*	......(1),
j k l	*e l m*	*d h j*	*d i m*	*d k n*	*f i l*	*e g k*	
m n p	*i k p*	*f g n*	*h l n*	*e h p*	*g j m*	*f j p*	
			a e j	*a f m*	*a h k*	*a i n*	
			b g p	*b i j*	*b e n*	*b d l*	
do.	do.	do.	*c h m*	*c d p*	*c g l*	*c f k*	......(2),
			d k n	*e g k*	*d i m*	*e h p*	
			f i l	*h l n*	*f j p*	*g j m*	

and the other four arrangements may be deduced from these two by the translation formula

a b c	*d e f*	*g h i*	*j k l*	*m n p*
a b c	*n m p*	*i g h*	*f d e*	*l k j*

It will be observed that the second form only differs from the first in the arrangement of twelve triads, appearing three in each of four walks.

The irregular solutions I have found are as follows :—

Type *a b c d e f g k n* ......(α)
h i j l m p ......(β) .

Two cases :

First case.

a b c	*a d g*	*a e j*	*a f m*	*a h k*	*a i n*	*a l p*
d e f	*b e n*	*b k p*	*b g j*	*b i m*	*b d l*	*b f h*
g h i	*c f k*	*c g m*	*c l n*	*c d p*	*c h j*	*c e i*
j k l	*h l m*	*d h n*	*d i k*	*e g l*	*e k m*	*d j m*
m n p	*i j p*	*f i l*	*e h p*	*f j n*	*f g p*	*g k n*

These triads can only be arranged in one way to give seven complete walks.

Second case.

a b c	*a d g*	*a h k*	*a l p*	*a e j*	*a f m*	*a i n*	
d e f	*b f k*	*b l n*	*b e h*	*b i m*	*b g j*	*b d p*	
g h i	*c e n*	*c f i*	*c g m*	*c d l*	*c k p*	*c h j*	......(1),
j k l	*h l m*	*d j m*	*d i k*	*f h p*	*d h n*	*e k m*	
m n p	*i j p*	*e g p*	*f j n*	*g k n*	*e i l*	*f g l*	

				a e j	*a f m*	*a i n*	
				b i m	*b d p*	*b g j*	
do.	do.	do.	do.	*c k p*	*c h j*	*c d l*	......(2);
				d h n	*e i l*	*e k m*	
				f g l	*g k n*	*f h p*	

and, by operating on these two sets twice in succession, by the translation formula

$$\begin{array}{c|c|c|c|c} a\ b\ c & d\ e\ f & g\ h\ i & j\ k\ l & m\ n\ p \\ c\ a\ b & e\ f\ d & n\ m\ p & i\ g\ h & l\ k\ j \end{array}$$

it will be found that identically the same triads are reproduced, and thus four other arrangements of these triads are obtained.

This second case is equivalent to one given by Woolhouse in his second paper. It will be observed that the second form in this case differs from the first in the arrangement of nine triads, appearing three together in three walks.

$$\text{Type}\quad \left.\begin{array}{l} a\ b\ c\ i\ k\ p \qquad\ \ldots\ldots(\alpha) \\ d\ e\ f\ g\ h\ j\ \ l\ m\ n \ldots\ldots(\beta) \end{array}\right\}.$$

Two cases, the first giving only one solution, and the second two solutions:

First case, giving one solution.

	1	δ	δ²	1	δ	δ²
a b c	*a d g*	*a e j*	*a f m*	*a h k*	*a l p*	*a n i*
d e f	*b h p*	*b l i*	*b n k*	*b e g*	*b f j*	*b d m*
g h i	*c j n*	*c m h*	*c g l*	*c f p*	*c d i*	*c e k*
j k l	*e l m*	*f n g*	*d h j*	*d l n*	*e n h*	*f h l*
m n p	*f i k*	*d k p*	*e p i*	*i j m*	*k m g*	*p g j*

This case may be arranged so as to show the cyclical property pointed out by Mr. Woolhouse, the letters following one another according to the table

$$\left.\begin{array}{l} c\ d\ f\ i\ e\ k\ p \\ b\ g\ m\ n\ j\ h\ l \end{array}\right\}.$$

Second case, giving two solutions.

	1	δ	δ^2	1	δ	δ^2	
a b c	*a h k*	*a l p*	*a n i*	*a d g*	*a e j*	*a f m*	
d e f	*b f g*	*b d j*	*b e m*	*b h p*	*b l i*	*b n k*	
g h i	*c j n*	*c m h*	*c g l*	*c e k*	*c f p*	*c d i*	......(1),
j k l	*d l m*	*e n g*	*f h j*	*f l n*	*d n h*	*e h l*	
m n p	*e i p*	*f k i*	*d p k*	*i j m*	*k m g*	*p g j*	
				a d g	*a e j*	*a f m*	
				b n k	*b h p*	*b l i*	
do.	do.	do.	do.	*c f p*	*c d i*	*c e k*	......(2);
				e h l	*f l n*	*d n h*	
				i j m	*k m g*	*p g j*	

the second form differing from the first only in the arrangement of nine of the triads. This case is equivalent to one given by Mr. Woolhouse in his second paper.

Type *a b c d e f g k m n p*......(α)
h i j l ......(β).

a b c	*a d g*	*a e j*	*a f m*	*a h k*	*a i n*	*a l p*
d e f	*b f k*	*b i p*	*b d n*	*b l m*	*b g j*	*b e h*
g h i	*c e p*	*c k n*	*c h j*	*c f i*	*c d l*	*c g m*
j k l	*h l n*	*d h m*	*e i l*	*d j p*	*e k m*	*d i k*
m n p	*i j m*	*f g l*	*g k p*	*e g n*	*f h p*	*f j n*

There is only one way of arranging this set of triads.

In conclusion, I would observe, that in no case have I succeeded in finding a solution in which the companions of any letter, *e.g.* *a*, are of the form (γ), and in all the solutions I have found the companions of all the elements of one row at least, *e.g.* *a b c*, are of the form (α).

The more or less symmetrical character of all these solutions is best seen by writing, side by side in three columns, the pairs of companions of *a b c*. It will seen that all these sets of companions consist of symmetrical arrangements of four triads *d e f*, *g j m*, *h l n*, *i k p*.

Other symmetrical arrangements can be formed, appearing at first sight to give new solutions; but, so far as I have been able to find, all these are, in fact, but simple translations of one or other of the typical forms already given; thus, instead of the four triads given above, we might have taken one or other of the following sets of four, viz.,

g h i, *d k n*, *e l m*, *f j p*
j k l, *e h p*, *d i m*, *f g n*
m n p, *f i l*, *d h j*, *e g k*

Take, for example, *g h i*, and translate it into *d e f*, making corresponding alterations in the other letters so as to keep the standard walk

and a's set of companions unaltered. We thus obtain a translation formula, which, when applied to the several typical forms, gives a fresh set of companions to b and c, and produces a form which in appearance differs from the typical form, of which, in fact, it is a mere translation.

Thursday, June 9th, 1881.

S. ROBERTS, Esq., F.R.S., President, in the Chair.

Mr. G. R. Dick, M.A., late Fellow of Caius College, Cambridge, Professor of Mathematics in the Royal College, Mauritius, was elected a member, and Prof. Mannheim and Mr. T. Craig were admitted into the Society.

The following communications were made:—

"Sur les surfaces parallèles," Prof. Mannheim. [Dr. Hirst proposed a vote of thanks to the author for his paper; this was carried by acclamation; M. Mannheim briefly responded; he also communicated a short "Note on the Wave Surface."]

"On certain Symbolic Operators," Mr. J. W. L. Glaisher.

"On a System of Coordinates," Prof. Genese.

"Note on a System of Cartesian Ovals passing through Four Points on a Circle," Mr. R. A. Roberts.

"On the Gaussian Theory of Surfaces," Prof. Cayley.

"On a Theorem in the Calculus of Operations," Mr. J. J. Walker.

"On Spherical Quartics, with a Quadruple Cyclic Arc, and a Triple Focus," Mr. H. M. Jeffery.

The following presents were made to the Library:—

"Educational Times," June, 1881.

"Beiblätter zu den Annalen der Physik und Chemie," Band v., Stücke 5, 6; 1881.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome xvi., 1, 2 Livraisons; Harlem, 1881.

"Atti della R. Accademia dei Lincei—Transunti," Vol. v., Fasc. 11, 12, 13, 14; Roma, 1881.

"Atti della R. Accademia dei Lincei—Memorie," Serie terza della Classe di Scienze Fisiche, Matematiche e Naturali, Vols. v., vi., vii., viii.; Roma, 1880.

"Atti della R. Accademia dei Lincei—Memorie," Serie terza della Classe di Scienze Morali, Storiche e Filologiche, Vol. iv., v.; Roma, 1881.

"Physical Society of London—Proceedings," Vol. iv., Pt. ii., Jan. to April, 1881.

"Sitzungsberichte der Physikalisch-medicinischen Societät zu Erlangen," 12 Heft, Nov. 1879 bis Aug. 1880.

"Smithsonian Report . . . . for the year 1879;" Washington, 1880.

"Bulletin de la Société Mathématique de France," Tome ix., No. 3; Paris, 1881.

"Proceedings of Royal Society," Vol. xxxii., Nos. 212, 213.

"Bulletin des Sciences Mathématiques et Astronomiques," Tome iv., Dec., 1880; Paris, 1880.

"Bulletin des Sciences Mathématiques et Astronomiques—Table des Matières et Noms d'Auteurs," Tome iv., deuxième Série, and Tome v., Janv., 1881.

"Monatsbericht," März, 1881.

"Ueber Lamé'sche Functionen," von F. Klein (pp. 213—246); and "Ueber Körper, welche von confocalen Flächen zweiten Grades begränzt sind," von F. Klein (pp. 410—427, Math. Annalen, Vol. xviii.).

On a System of Coordinates. *By* Prof. Genese, M.A.

[*Read June* 9*th*, 1881.]

1. Let APB connote a *positive* rotation from the direction PA to the direction PB, so that

$$APB+BPA = 2\pi.$$

If A, B be points on an axis OI, I being at infinity,

$$IBP-IAP = APB \text{ (Fig. 1)},$$

or $\quad APB-2\pi$ (Fig. 2).

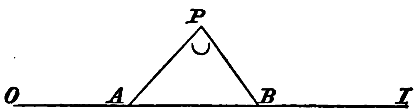


Fig. 1.

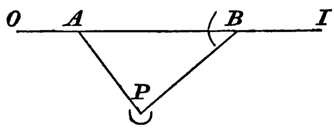


Fig. 2.

Now, taking O as origin, OI as axis of x, let x, y be the rectangular coordinates of P, and let a, b, c be the abscissæ of three points A, B, C on OI. Then

$$\cot IAP = \frac{x-a}{y}, \text{ \&c.},$$

therefore $\quad \cot APB = \dfrac{(x-a)(x-b)+y^2}{(b-a)\,y}$, $\cot BPC = \text{\&c.}$ ....(1).

Let θ, ϕ, ψ denote the angles BPC, CPA, APB respectively, and λ, μ,

ν the cotangents of these angles; then

$$\theta+\phi+\psi = 2\pi, \text{ and } \mu\nu+\nu\lambda+\lambda\mu = 1 \quad \ldots\ldots\ldots\ldots(2).$$

The object of this paper is to exhibit results obtained by taking λ, μ, ν as the coordinates of P.

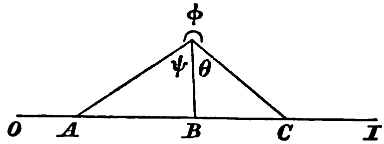


Fig. 3.

2. Solving for x^2+y^2, x, and y, in terms of λ, μ, ν,

$$\left.\begin{aligned} x^2+y^2 &= \frac{a^2(c-b)^2\lambda+\&c.}{(c-b)^2\lambda+\&c.} \\ x &= \frac{a(c-b)^2\lambda+\&c.}{(c-b)^2\lambda+\&c.} \\ y &= \frac{-(c-b)(a-c)(b-a)}{(c-b)^2\lambda+\&c.} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(3).$$

And we notice that the degree of any locus in λ, μ, ν is not higher than its degree in x, y; but that curves passing through the circular points at infinity will be represented by equations of lower degree in λ, μ, ν than in x, y. On the other hand, we have one more coordinate, and the relation $\mu\nu+\nu\lambda+\lambda\mu = 1$; perhaps, however, analogy may suggest properties of the surface $xy+yz+zx = 1$.

3. The equation of the first degree,

$$A\lambda+B\mu+C\nu = D \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

represents in general a circle.

But if $\dfrac{A}{c-b}+\dfrac{B}{a-c}+\dfrac{C}{b-a} = 0$, then (4) will represent a straight line.

The quantities $c-b$, $a-c$, $b-a$,—that is, with the usual sign-convention BC, CA, AB,—occurring frequently, it will be convenient to denote them by c_1, c_2, c_3. Thus, the condition that (4) should represent a straight line is

$$\frac{A}{c_1}+\frac{B}{c_2}+\frac{C}{c_3} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

4. If $A\lambda+B\mu+C\nu = D$ represent a straight line PT cutting the axis of x in T, and $ITP = \alpha$,

$$A : B : C : D :: c_1^2\,AT : c_2^2\,BT : c_3^2\,CT : c_1c_2c_3 \cot\alpha \ldots\ldots(6);$$

and this establishes a relation between λ, μ, ν and *line-coordinates* referred to the intercepts p, q, r from ordinates through A, B, C, viz.,

$$\frac{A}{D} = \frac{c_1 p}{c_2 c_3},\quad \frac{B}{D} = \&c. \quad \ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Cor. 1.—The straight line is an ordinate to the axis if

$$D = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Cor. 2.—Two straight lines,

$$A\lambda+B\mu+C\nu = D,$$
$$A'\lambda+B'\mu+C'\nu = D',$$

are parallel, if $$\frac{A-A'}{c_1^2} = \frac{B-B'}{c_2^2} = \frac{C-C'}{c_3^2} \quad\text{.......................(9).}$$

The equation to the straight line joining two points $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$ is

$$\begin{vmatrix} \lambda, & \mu, & \nu, & 1 \\ \lambda_1, & \mu_1, & \nu_1, & 1 \\ \lambda_2, & \mu_2, & \nu_2, & 1 \\ \frac{1}{c_1}, & \frac{1}{c_2}, & \frac{1}{c_3}, & 0 \end{vmatrix} = 0 \quad\text{.......................(10).}$$

The equation to a straight line through B is

$$c_1\lambda - c_3\nu = c_2 \cot a \quad\text{.........................(11),}$$

a relation enabling us to change from λ, μ, ν to biangular coordinates.

The equation $$c_1^2\lambda + c_2^2\mu + c_3^2\nu = D \quad\text{.........................(12)}$$

represents the same as $$y = \frac{-c_1c_2c_3}{D},$$

i.e., a straight line parallel to axis of x.

If $D = 0$, $y = \infty$, thus

$$c_1^2\lambda + c_2^2\mu + c_3^2\nu = 0 \quad\text{.........................(13)}$$

represents the straight line at infinity.

Again, $$a\, c_1^2\lambda + b\, c_2^2\mu + c\, c_3^2\nu = D \quad\text{.....................(14)}$$

represents a straight line through O, and, in the case of $D=0$, it is the axis of y.

The Circle.

5. When $\frac{A}{c_1} + \frac{B}{c_2} + \frac{C}{c_3}$ is not zero, the equation

$$A\lambda + B\mu + C\nu = D$$

represents a circle.

By considering the intersections T, T' with OI, for which λ, μ, ν each $= \infty$, but are in the ratios $\frac{1}{\sin\theta} : \frac{1}{\sin\phi} : \frac{1}{\sin\psi}$, or as

$$\frac{1}{c_1 AT} : \frac{1}{c_2 BT} : \frac{1}{c_3 CT},$$

we find that $A : B : C :: c_1^2 AT\,.\,AT' : c_2^2 BT\,.\,BT' : c_3^2 CT\,.\,CT'$......(15).

Also it may be shown that

$$A : D :: c_1^2 AT\,.\,AT' : c_1c_2c_3\,.\,2\beta \quad\text{.................(16),}$$

where β is the ordinate of the centre.

The coordinates α, β of the centre of the circle are

$$\left.\begin{aligned}\alpha &= \frac{(b+c)\dfrac{A}{c_1}+\&c.}{2\left(\dfrac{A}{c_1}+\&c.\right)}\\ \beta &= \frac{D}{2\left(\dfrac{A}{c_1}+\&c.\right)}\\ \text{and}\qquad \text{radius} &= \frac{\sqrt{D^2+A^2+\&c.-2BC-\&c.}}{2\left(\dfrac{A}{c_1}+\&c.\right)}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(17).$$

From the geometrical interpretations of ABC, we see that the equation to any circle through B is $A\lambda+C\nu = D$ ....(18);
also that, if this circle touch the axis at B, the equation becomes

$$\lambda+\nu = k \quad\ldots\ldots\ldots\ldots(19).$$

The equation to a circle through A and C is, of course,

$$\mu = d \quad\ldots\ldots\ldots\ldots(20).$$

The condition that the last-mentioned circles should touch is [remembering that $\lambda\nu+\mu(\lambda+\nu) = 1$]

$$k^2 = 4(1-kd) \quad\ldots\ldots\ldots\ldots(21).$$

This condition leads to an easy proof of Feuerbach's Theorem that the nine-point circle touches the inscribed and escribed circles;—at the point of contact $\lambda = \nu$ (Wolstenholme, Prob. 43).

The equation $\lambda = 0$, *i.e.* $\cot\theta = 0$, represents the circle on BC as diameter.

6. *The equation to a circle having its centre on the axis* is

$$A\lambda+B\mu+C\nu = 0 \quad\ldots\ldots\ldots\ldots(22).$$

This will reduce to a point-circle if

$$A^2+\&c.-2BC-\&c. = 0,$$

or

$$\sqrt{A}+\sqrt{B}+\sqrt{C} = 0 \quad\ldots\ldots\ldots\ldots(23).$$

In particular, $\mu+\nu = 0$ represents a point-circle at A ....(24).

The equation

$$\begin{vmatrix}\lambda, & \mu, & \nu\\ \lambda_1, & \mu_1, & \nu_1\\ \lambda_2, & \mu_2, & \nu_2\end{vmatrix} = 0 \quad\ldots\ldots\ldots\ldots(25)$$

represents the circle passing through the points $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$, ıd having its centre on the axis ABC.

If P_1, P_2 be these points, P_1', P_2' their images with respect to the axis ABC, the circle passes through $P_1P_2P_1'P_2'$.

For brevity, such a circle will hereinafter be called the axial-circle (P_1P_2), or simply (P_1P_2).

The Equation of the Second Degree.

7. The equation $(A, B, C, D, F, G, H, L, M, N)(\lambda, \mu, \nu, 1)^2 = 0$ represents the general bicircular quartic.

In the present communication, the homogeneous equation

$$(A, B, C, F, G, H)(\lambda, \mu, \nu)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(26)$$

only will be considered.

Its equivalent in x, y coordinates is

$$l(x^2+y^2)^2+(mx+n)(x^2+y^2)+px^2+qx+r = 0 \ldots\ldots\ldots(27).$$

Thus it may represent a *bicircular quartic with collinear foci*, a *circular cubic*, a *conic*, *two circles*, &c.

8. Considering the equations (22) and (25), we see that the analytical processes which demonstrate the non-metrical properties of the straight line and conic, may be interpreted, *mutatis mutandis*, to give properties of axial-circles and the curves represented by (26).

Ex. 1.—Two axial-circles may be drawn through any point P in a plane to touch any such curve at Q, R, say. The axial-circle (QR) may be called the polar of P. Then, if P move on a fixed axial-circle, the polar of P will pass through a fixed point (and its image).

Ex. 2.—From the analysis of Pascal's Theorem, if P, Q, R, S, T, U be points on a $(\lambda, \mu, \nu)^2$ curve, the axial-circles (PQ) and (ST), (QR) and (TU), (RS) and (UP) intersect in three points (and their images) lying on an axial-circle.

A closer study of the (λ, μ, ν) system may, perhaps, show a connection between the metrical properties of conics and $(\lambda, \mu, \nu)^2$ curves.

On the Generation of $(\lambda, \mu, \nu)^2$ Curves.

9. Let A_1, A_2 be two fixed points in the plane xy, C_1, C_2 two variable points on the axis of x, P the intersection of C_1A_1, C_2A_2; then, a *suitable* (2, 2) correspondence being found for C_1, C_2, P may be made to describe any conic.

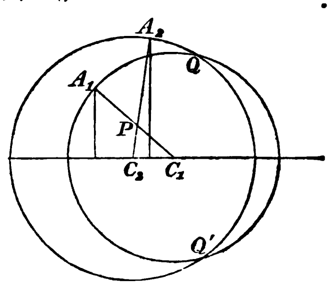


Fig. 4.

By analogy, if circles be described with centres C_1, C_2, passing through A_1, A_2 respectively, and a *suitable* (2, 2) correspondence be found for C_1, C_2, the

intersections Q, Q' of the circles will describe any given $(\lambda, \mu, \nu)^2$ curve.*

10. If a_1, a_2 be the distances of C_1, C_2 from the intersection of A_1, A_2 with the axis of x, it will be found that the most general correspondence which will make P describe a conic is

$$(lmn, pqr)\left(\frac{1}{a_1}, \frac{1}{a_2}, 1\right)^2 = 0 \quad \text{..................(28)}.$$

Analogously, if a_1, a_2 be the distances of C_1, C_2 from the centre of the axial circle $(A_1 A_2)$, the same correspondence will make Q_1, Q_2 describe the general $(\lambda, \mu, \nu)^2$ curve.

11. If we put $l = 0$, $m = 0$, we get a (1, 1) correspondence; the locus of P is either a conic *passing through* A_1, A_2, or a straight line, and the locus of Q, Q' will be a $(\lambda, \mu, \nu)^2$ curve *passing through* A_1, A_2.†

12. Now, let A_1, A_2 be points of a $(\lambda, \mu, \nu)^2$ curve *lying on the axis*, then the following apparatus will trace the curve—

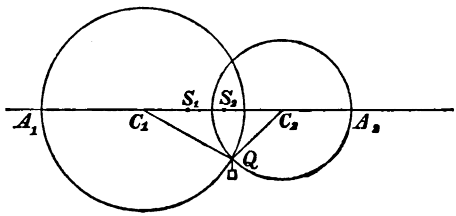


Fig. 6.

Two strings fastened to a pencil at Q with a weight attached, pass round C_1, C_2, and are fastened to fixed pegs S_1, S_2 in an axis. C_1, C_2 being made to move on this axis homographically, Q describes one of the $(\lambda, \mu, \nu)^2$ curves (S_1A_1, S_2A_2 are, of course, the lengths of the strings).

---

* It is to be observed that the general (2, 2) correspondence between C_1 and C_2 would make P describe a quartic having A_1 and A_2 for double points. For such a quartic *would* give a (2, 2) correspondence, and the degree n of the locus of P cannot be greater than 4, because then A_1, A_2 would be two multiple points of the $(n-2)^{\text{th}}$ order.

Also, a suitable (2, 2) correspondence will make P describe ***any*** given quartic having A_1, A_2 as double points. For eight other points completely determine the quartic, and eight positions of the pair C_1, C_2 determine the (2, 2) correspondence.

† Such a correspondence may be easily constructed practically.

Let F_1, F_2 be the foci of the given homography, I_1, I_2 the centres, *i.e.*, the positions of C_1 and C_2 when C_2 and C_1 are at infinity. Draw ordinates I_1B_1, I_2B_2 so that

$$(I_1B_1)^2 = (I_2B_2)^2 = I_1F_1 \cdot I_1F_2.$$

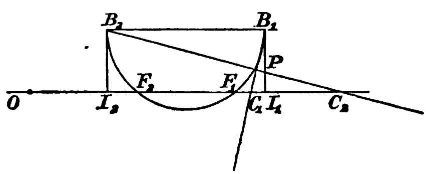


Fig. 5.

Then, if P be any point on a circle described on B_1B_2 as diameter, B_1P, B_2P determine corresponding positions of C_1, C_2; thus, ***if two rods rigidly attached at right angles be made*** to move round fixed pegs (B_1, B_2) suitably placed, they will determine ***any given*** homography on OI.

In particular, if C_1C_2 be a fixed length, Q will describe a central conic, whose eccentricity $= \sqrt{\dfrac{2C_1C_2}{A_1A_2}}$ ....(29).*

If $C_1C_2 = A_1A_2$, $e = \sqrt{2}$; and Q traces out a rectangular hyperbola.

In this case, the apparatus may be simplified, for $C_1Q = C_1A_1 = C_2A_2 = C_2Q$. Thus, if one arm of a rigid right angle C_1MQ slide on a horizontal axis, and a weight Q, attached to a string which passes over a pulley at C_1 (moving with C_1M) and is fastened to a fixed peg S_1 in the axis, be allowed to fall down MQ (vertical), Q will describe a rectangular hyperbola.

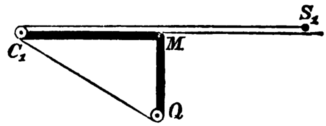


Fig. 7.

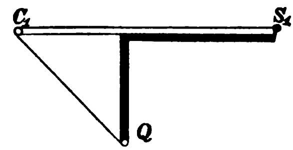


Fig. 8.

If the pulley C_1 be fixed to the axis, and S_1 move with the rigid angle, Q will trace out a parabola.

The $(\lambda, \mu, \nu)^2$ Curve considered as an Envelope.

If Q, R be two points on fixed straight lines a_1, a_2 respectively, and a suitable (2, 2) correspondence be established between Q and R, the join QR will envelope a conic.

Analogously, if Q, R move on two fixed axial circles a_1, a_2, the axial circle (QR) will, with a suitable (2, 2) correspondence, envelope a $(\lambda, \mu, \nu)^2$ curve.

14. If we substitute a (1, 1) correspondence, the envelope will touch a_1, a_2 in each case.

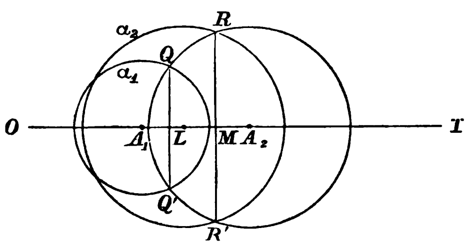


Fig. 9.

One mode of establishing such a correspondence is as follows :—

Let the radical axis of (QR) and a_1 cut OI at L, and let M be the

* This theorem may be demonstrated geometrically at once from the following,—"If M be the middle point of any chord AQ, and from M perpendiculars be drawn to the chord and the axis, meeting the axis respectively in G and N; then $CG = e^2 . CN$, C being the centre."

corresponding point for a_2; if L, M be homographically related, the envelope of the circle QR will be a $(\lambda, \mu, \nu)^2$ curve, touching a_1 and a_2.

In particular, *if LM be of constant length, the envelope will be a conic.*

It is remarkable that the eccentricity of this conic is

$$\sqrt{\frac{A_1 A_2}{2LM}} \text{ ....} (30),$$

where A_1, A_2 are the centres of a_1, a_2, *i.e.*, the inverse of the form (29) for the conic as a locus.

If the correspondence take the form

$$\frac{l}{OL} + \frac{m}{OM} = 1,$$

the circle (QR) will pass through a fixed point.

15. Again, let the circles be replaced by two straight lines AQ, BR perpendicular to the axis OAB.

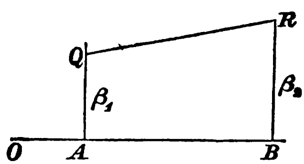


Fig. 10.

Then, a suitable (1, 1) correspondence being established between $AQ^2 = \beta_1^2$ and $BR^2 = \beta_2^2$, the axial circle (QR) will envelope a $(\lambda, \mu, \nu)^2$ curve touching AQ, BR. The general relation is between β_1^2, β_2^2 and not β_1, β_2, because the axial circles pass through the images of Q, R in the axis, and are equally well determined by $AQ = -\beta_1$, $BR = -\beta_2$.

(If we consider anallagmatic surfaces instead of curves, β_1^2, β_2^2 are proportional to the areas of the traces made by a movable sphere on two fixed planes.)

For the particular case $l\beta_1^2 + m\beta_2^2 = n$, the circle (QR) passes through two fixed points.

16. It may, however, be added that the correspondence $l\beta_1 + m\beta_2 = n$, for which the straight line QR passes through a fixed point, gives a circular cubic as the envelope of (QR). In particular, if $\beta_2 \pm \beta_1 =$ constant, the envelope is an ellipse having AB for minor axis; with upper sign, the join QR passes through the focus, with lower sign, QR is constant and equal to the major axis.

17. In order that the axial-circles should have a real envelope, consecutive members must intersect.

Let a, $a + da$ be the abscissæ of the centres of axial-circles of radii N, $N + dN$. By Euc. I. 20, the condition of intersection is $dN < da$ numerically. In fact,

$$\frac{dN}{da} = -\cos\psi \text{ ....} (31),$$

where ψ is the angle which N makes with OI.

The Foci on the Axis.

18. The condition that the axial-circle

$$l\lambda+m\mu+n\nu=0$$

should touch $(A, B, C, F, G, H)(\lambda, \mu, \nu)^2=0$,

is $l^2(BC-F^2)+\&c.+2mn(GH-AF)+\&c.=0$ ....(32).

Now the condition that the circle should reduce to a point is

$$l^2+\&c.-2mn-\&c.=0.$$

The two relations give, in general, four values of the ratios $l : m : n$, that is, *there are four foci on the axis.*

Cartesian Ovals.

19. If r_1, r_2, r_3 be the distances of any point P from A, B, C, we find

that $r_1 : r_2 : r_3 :: \dfrac{\sin\theta}{c_1} : \dfrac{\sin\phi}{c_2} : \dfrac{\sin\psi}{c_3}$ ....(33).

Thus the equation $lc_1r_1+mc_2r_2+nc_3r_3=0$

is equivalent to $l\sin\theta+m\sin\phi+n\sin\psi=0$

or $$\frac{l}{\sqrt{1+\lambda^2}}+\frac{m}{\sqrt{1+\mu^2}}+\frac{n}{\sqrt{1+\nu^2}}=0.$$

But $\lambda^2+1=\lambda^2+\lambda\mu+\lambda\nu+\mu\nu=(\lambda+\mu)(\lambda+\nu)$.

Thus our equation is

$$l\sqrt{\mu+\nu}+m\sqrt{\nu+\lambda}+n\sqrt{\lambda+\mu}=0 \text{ ....(34)};$$

as we should expect, for the point-circles $(\mu+\nu)=0$, $\nu+\lambda=0$, $\lambda+\mu=0$, at A, B, C, are foci.

It is easily shown that $l^2c_1+m^2c_2+n^2c_3=0$ ....(35)

is the *condition that the fourth focus should be at infinity.*

With this condition, $l\sqrt{\mu+\nu}+\&c.=0$

represents a Cartesian oval.

20. The condition that

$$p(\mu+\nu)+q(\nu+\lambda)+r(\lambda+\mu)=0$$

should touch the Cartesian is

$$\frac{l^2}{p}+\frac{m^2}{q}+\frac{n^2}{r}=0.$$

Now, returning to x, y coordinates, we find that

$$\mu+\nu : \nu+\lambda : \lambda+\mu :: c_1^2\{(x-a)^2+y^2\} : \&c. \text{............(36)}.$$

Therefore $p(\mu+\nu)+\&c. = 0$ represents the circle

$$L(x^2+y^2-2ax+t) = 0,$$

where

$$pc_1^2 + qc_2^2 + rc_3^2 = L,$$

$$pc_1^2 a + \&c. = La,$$

$$pc_1^2 a^2 + \&c. = Lt,$$

whence

$$pc_1^2 = \frac{L(bc-\overline{b+c}\,a+t)}{-c_2c_3}, \quad qc_2^2 = \&c.$$

If we replace t by a^2-N^2, N being the radius of the circle,

$$pc_1 : qc_2 : rc_3 :: (a-b)(a-c)-N^2 : \&c.$$

Thus, if N be the length of a normal to a Cartesian drawn from the point G on the axis $(OG = a)$,

$$\frac{l^2c_1}{(a-b)(a-c)-N^2} + \frac{m^2c_2}{(a-c)(a-a)-N^2} + \&c. = 0;$$

or, since $l^2c_1+m^2c_2+n^2c_3 = 0$,

$$N^2 = \frac{(a-a)(a-b)(a-c)}{a+\dfrac{l^2c_1bc+m^2c_2ca+n^2c_3ab}{l^2c_1a+m^2c_2b+n^2c_3c}} \quad \text{.................(37).}$$

If the origin O be so chosen that

$$l^2c_1bc+\&c. = 0,$$

then

$$N^2 = \frac{AG\,.\,BG\,.\,CG}{OG} \quad \text{........................(38).}$$

For the determination of O, we get

$$\frac{OA}{l^2} = \frac{OB}{m^2} = \frac{OC}{n^2} \quad \text{..........................(39).}$$

i.e., it is the triple focus, and it is clearly the point in which *the asymptote to the evolute of the Cartesian cuts the axis.*

21. Since $\dfrac{dN}{da} = -\cos\psi$, ψ being the angle the normal makes with the axis of x, we get

$$-\cos\psi = \frac{2a-(a+b+c)+\dfrac{abc}{a^2}}{2\sqrt{\dfrac{(a-b)(a-c)(a-a)}{a}}} \quad \text{..............(40),}$$

which may be regarded as the tangential equation to the evolute of the Cartesian.

22. A simpler formula is obtainable from (38), if we take logarithms

of both sides before differentiation, viz.,

$$\frac{2}{N}(-\cos\psi) = \frac{1}{AG} + \frac{1}{BG} + \frac{1}{CG} - \frac{1}{OG};$$

or, if T be the point at which the tangent corresponding to N meets the axis,

$$\frac{2}{TG} = \frac{1}{AG} + \frac{1}{BG} + \frac{1}{CG} - \frac{1}{OG} \quad \text{..................(41)},$$

and this is easily changed into

$$\frac{OT}{OG} = \frac{AT}{AG} + \frac{BT}{BG} + \frac{CT}{CG} \quad \text{.....................(42).*}$$

Inversion.

The formulæ for inversion in λ, μ, ν coordinates are very simple, viz., if (λ', μ', ν') be the inverse of (λ, μ, ν) with respect to any centre O on the axis, the radius of inversion being taken $= \sqrt{OA \,.\, OC}$, then

$$\mu' = \mu,$$

$$\lambda' = \left(\frac{r}{p} - 1\right)\mu + \frac{r}{p}\nu,$$

$$\nu' = \left(\frac{p}{r} - 1\right)\mu + \frac{p}{r}\lambda,$$

where $p = c_1^2 \,.\, OA, \quad r = c_3^2 \,.\, OC.$

In particular, if $OA : OC :: c_3^2 : c_1^2$ (or, which is the same thing, $OB^2 = OA \,.\, OC$), then

$$\mu' = \mu,$$

$$\nu' = \lambda,$$

$$\lambda' = \nu.$$

Note on Apparatus for describing Conics.

If C_1, C_2 be made to move with uniform velocity u, then Q will have a constant parallel velocity $= \frac{A_1A_2'}{C_1C_2}u$, and therefore (Newton, Prop. viii.) a vertical acceleration varying inversely as the cube of the distance from A_1A_2.

It will be observed that only a part of the curve can be described, the apparatus failing to work when the angle QC_1C_2, or QC_2C_1, be-

* If C be taken at infinity, the system of coordinates reduces to a form of biangular coordinates: $\theta = \beta$, $\phi = \pi + \alpha$, and $\psi = \beta - \alpha$. Also, $\sin\theta : \sin\phi : \sin\psi :: r_1 : -r_2 : c_3$. Thus equation (34) will represent the curve $lr_1 - mr_2 + nc_3 = 0$. When $l^2 = m^2$, this represents a central conic; the point O is at infinity, and equation (42) becomes $0 = \frac{AT}{AG} + \frac{BT}{BG}$, and expresses that the tangent and normal to a conic divide the line joining the foci harmonically.

comes a right angle. This remark does not apply to the modification for the rectangular hyperbola or to the apparatus for the parabola.

Additional Note.

I have found, since writing the above, that the method of Art. 10, *as regards the* (1, 1) *correspondence*, is not new. It is used in Dr. Casey's elegant memoir on "Bicircular Quartics," Art. 10, and was originally given by Chasles, "Comptes Rendus," 1853. The extension, by using curves of Index 1 in place of conics passing through four points, is due to M. Terquem; and not, as stated by Clebsch (p. 376, "Vorlesungen über Geometrie") to M. de Jonquières. See "Théorèmes segmentaires," *Nouvelles Annales*, 1853, p. 358.

To prevent misconception I have to add, respecting the system of coordinates, that I had in the first instance reinvented Mr. Walton's "Trigonic Coordinates," *Quarterly Journal*, Vol. ix., p. 340. I suppressed my paper on the subject, and went on with the work only on perceiving the advantages mentioned in § 2.

On Spherical Quartics, with a Quadruple Cyclic Arc, and a Triple Focus. *By* Henry M. Jeffery.

[*Read June 9th*, 1881.]

1. In general, quartics with quadruple cyclic arcs have also quadruple foci in their quadrantal poles; but if a cyclic arc of the satellite-conic coincide with the quadruple cyclic arc of the quartic, its quadrantal pole is a triple focus. (§ 11.)

Such a group of spherical quartics may be thus defined:

$$\kappa = (1+dx)(1+x^2+y^2)+\lambda(1+x^2+y^2)^2 \quad \ldots\ldots\ldots\ldots\ldots \text{(A)},$$

if x, y denote by Gudermann's system of spherical coordinates the tangents of arcs, intercepted on the arcs of coordinates by great circles drawn from their quadrantal poles.*

These quartics are of the eighth class, if they are non-singular, and may have five single foci, collinear with the triple focus. If the quartics are nodal, two single foci unite, as *in plano*, at a node and disappear; such quartics are of the sixth class (§ 12.)

If $d^2 < 1$, there may be two cusped quartics in the group, pear-shaped (Fig. 1), and cardioidal (Fig. 2); these are of the fifth class.

---

* If α, β be arcs drawn perpendicularly from any point of a quartic to the cyclic arcs, their geometrical relation is thus expressed:

$$\kappa(\sin\alpha)^4 = \sin\alpha\sin\beta+\lambda \quad \ldots\ldots\ldots\ldots\ldots \text{(A)}.$$

If $d^2 = 1$, $\lambda = 1$, $\kappa = \frac{1}{4}$, the corresponding point-quartic has only one single focus, and is of the fourth class.

2. All the quartics of this group are non-folium, unifolium, or bifolium, *i.e.*, have none, one, or two depressions in one oval, symmetrical with respect to their axes. These folia, which are characterised by two points of inflexion and a bitangent, may terminate, by varying the parameters (κ, λ) of the group, either in a singular quartic or in a folium-point, *i.e.*, a point of undulation. These terminal values of the parameters (κ, λ) may be exhibited in their mutual relation by two discriminating plane curves. (Figs. 5–8.)

The various alterations and transitions in the forms of the quartics are thus explained in detail in § 5.

3. To exhibit the mutual relation between the parameters κ and λ, when the spherical quartics of this group are nodal, as the first discriminating curve. (Fig. 5.)

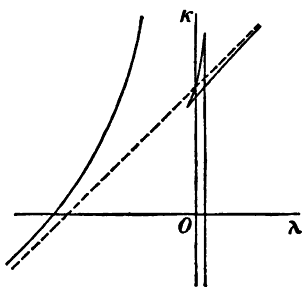


Fig. 5.

If $\phi = 0$ be the equation to the group (A), the conditions of singularity are

$$\frac{d\phi}{dx} = 0, \quad \frac{d\phi}{dy} = 0.$$

These are

(1)... $4\kappa x = 2x(1+x^2) - d(1-x^4)$, $y = 0$.

(2)... $4\lambda x(1+x^2) + 2x + d(1+3x^2) = 0$.

The eliminant is a sextic, which may be obtained directly as the discriminant of the quartic (A), when $y = 0$, and two apses coincide at a node.

$$\left\{\lambda(\lambda-\kappa+1) - \frac{d^2}{4} + \frac{1}{12}(2\lambda+1)^2\right\}^3$$

$$= 27\left\{\frac{1}{6}\lambda(\lambda-\kappa+1)(2\lambda+1) + \frac{1}{48}d^2(2\lambda+1) - \frac{\lambda d^2}{16} - (\lambda-\kappa+1)\frac{d^2}{16} - \frac{1}{216}(2\lambda+1)^3\right\}.$$

The highest terms of this sextic are a multiple of $\kappa^2\lambda^3(\lambda-\kappa)$.

But this unicursal curve is more conveniently drawn from the implicit equations (1) and (2).

It has two asymptotes, the (κ) axis, and the line $(\kappa = \lambda+1)$. If $d^2 < 1$, it has a stapete, but is smooth if $d^2 > 1$.

For $\frac{d\kappa}{dx} = 0$, $\frac{d\lambda}{dx} = 0$, at a singular point; in each case

$$3dx^4 + 4x^3 + d = 0.$$

The discriminant of this quartic is $d^4 - 1$.

If $d^2 = 1$, the sextic has a triple or stapete-point. This stapete is important, since the critical quartics of the given group only occur (see § 5) for values of (κ, λ) thereon.

The other limit of the stapete is thus found:

As d approaches zero, $x = -\frac{4}{3d}$, and $-\left(\frac{d}{4}\right)^{\frac{1}{3}}$.

The corresponding cusps (κ, λ) are $(\infty, 0)$, and $(\frac{1}{2}, -\frac{1}{2})$.

One cusp of the stapete is at the end of the (κ) line of reference, the other at a finite distance from the origin.

In the limit $(d = 0)$, the quartic becomes two concentric small circles, which unite in a double circle, when $4\kappa\lambda + 1 = 0$. When $\kappa = \frac{1}{2}$, $\lambda = -\frac{1}{2}$, at an apse of this hyperbola, or when $\kappa = \infty$, $\lambda = 0$, the quartic terminates in a double point-circle.

4. To exhibit the mutual relation which exists between the parameters, when the spherical quartics of this group have a folium-point (or point of undulation), as the second discriminating curve. (Fig. 6.)

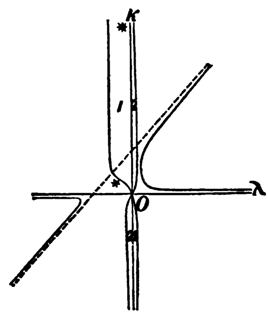


Fig. 6.

The conditions of such a point are in spherics, as *in plano*,

$$\frac{d^2y}{dx^2} = 0, \quad \frac{d^3y}{dx^3} = 0.$$

If $y^2 = u$, these conditions take the form

$$\frac{d^3u}{dx^3} = 0, \quad 2u\frac{d^2u}{dx^2} = \left(\frac{du}{dx}\right)^2.$$

Since the equation to the quartic group (A) is

$$\kappa = (1+dx)(1+x^2+y^2) + \lambda(1+x^2+y^2)^2,$$

these conditions lead to the relations

$$1 + dx = 0,$$

$$2\left\{-x^2 - 1 \pm \sqrt{\frac{\kappa}{\lambda}}\right\}\left\{-2 \pm \frac{d^2}{4\sqrt{(\kappa\lambda^3)}}\right\} = \frac{1}{4\lambda^2}(4\lambda x + d)^2,$$

whence the quintic is obtained

$$\kappa(16\lambda^2 + 8\lambda + d^2)^2 = 4\lambda(8\kappa\lambda + d^2 + 1)^2.$$

One condition, that κ have equal roots, is

$$(16\lambda^2 + 8\lambda + d^2)^2 = 128\lambda^2(d^2 + 1).$$

This quartic, again, has equal roots when $d^2 = 1$; the corresponding curve (Fig. 6) unites two branches in a tacnode, at which the tangent is parallel to the (κ) axis.

It should be remarked that the ratio $\frac{\kappa}{\lambda}$ is positive, except when

$$\left.\begin{array}{l}(1)\ldots 16\lambda^2+8\lambda+d^2=0\\ (2)\ldots 8\kappa\lambda+d^2+1 \quad =0\end{array}\right\},$$

when it is indeterminate. The acnodes for these values of (κ, λ) are denoted by asterisks. There are two, one, or no acnode, depending on the values of λ in (1). When λ has equal roots, *i.e.*, when $4\lambda+1=0$ and $\kappa=0$, two separate acnodes unite in a single acnode.

[Otherwise, if the equation to the group (A) be transformed to polar coordinates by the formulæ

$$x=\frac{1}{u}\cos\theta,\quad y=\frac{1}{u}\sin\theta,\quad \text{where}\quad \frac{1}{u}=\tan r,$$

it becomes
$$\frac{\kappa u^4}{1+u^2}-\lambda(1+u^2)=u^2+ud\cos\theta.$$

Differentiate thrice, and observe the conditions for a folium-point or point of undulation $\frac{d^2u}{d\theta^2}+u=0,\quad \frac{d^3u}{d\theta^3}+\frac{du}{d\theta}=0.$

The resulting conditions appear as two factors

$$\left(\frac{du}{d\theta}\right)^2=\frac{u^2(1+u^2)}{1-u^2},\quad \frac{du}{d\theta}=0.$$

The first factor gives the curve already discussed.

From the second factor $\left(\frac{du}{d\theta}=0\right)$, $\sin\theta=0$, and

$$\frac{\kappa u^4}{1+u^2}=-\lambda(1+u^2)=\tfrac{1}{2}(u^2+ud).$$

This curve is the serpentine portion of Fig. 6, which occurs only in the second and fourth quadrants.

The first (Fig. 5) and second (Fig. 6) discriminating curves touch at infinity.

This occurs when both $\frac{du}{d\theta}=0$ and $\left(\frac{du}{d\theta}\right)^2=\frac{u^2(1+u^2)}{1-u^2}$.]

5. All spherical quartics of this group may be exhibited by the aid of these two discriminating curves.

First, the critical quartics are considered.

The cases are distinguished as $d^2 > = < 1$, *i.e.*, as the cyclic arc of the satellite-conic is distant above 45°, or less than 45° from the quadruple cyclic arc.

In the first two cases there is no critical quartic when (κ, λ) is a point on the first discriminating curve (Fig. 5), except in the terminal form of two coincident points; if (κ, λ) lie beyond the hyperbolic branches, no quartic is possible.

If $d^2 < 1$, there are critical quartics, for every value of (κ, λ) in the stapete of the curve. (Fig. 5.) If (κ, λ) is its upper cusp, the corresponding quartic is a pirum, or cusped quartic. For values of (κ, λ) on the stapete, near its upper cusp, there correspond lemniscatoid, crunodal, and acnodal quartics. (Fig. 1.)

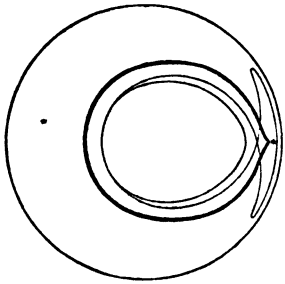


Fig. 1.

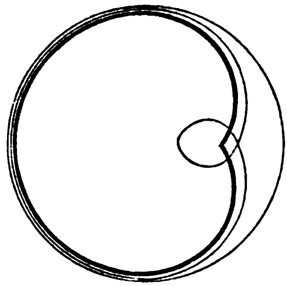


Fig. 2.

If (κ, λ) is the lower cusp of the stapete, according as $d^2 > = < \frac{1}{3}$, the corresponding quartic is cardioidal, or a cissoidal cubic, or a second reversed pirum, whose companion-nodal curves are limaçonoid or lemniscatoid, crunodal, and acnodal. (Fig. 2.) The forms vary as the lower cusp is to the right of the (κ) axis, upon it, or to its left.

Secondly, the quartics of the group, which have a folium-point, or point of undulation, are determined by the second discriminating curve.

[We are concerned with the serpentine portion only of Fig. 6, since no quartic corresponds to values of (κ, λ) in the outer hyperbolic branches. The quartics, whose parameters are the coordinates of points lying between the serpentine branches of Fig. 6 are bifolium in the first and third, and unifolium in the second and fourth quadrants. The number of stapetes is distinguished by the (κ) axis.] If $\lambda = 0$, the quartic is resolved into a cubic and $1 = 0$, the quadrantal polar of the quadruple focus at the origin. The critical and companion curves are exhibited in Fig. 3. If λ is negative, the quartics are unifolium, whose

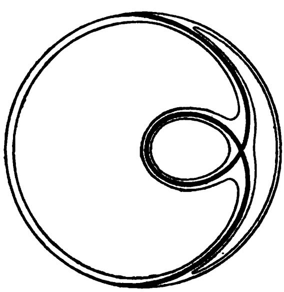


Fig. 3.

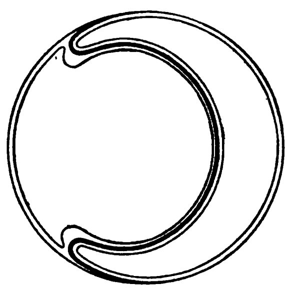


Fig. 4.

forms are terminated in point-quartics at the first discriminating curve.

These changes are exhibited in Fig. 4, where it will be understood that, if λ be positive, the apse is reversed, and the quartic is bifolium.

These figures are orthographically projected on the equatorial plane of the quadruple focus.

6. On Equable Spherical Spirals.

The critical cubic, which corresponds to the values 0, 1, $-\frac{1}{2}$ of the parameters λ, κ, d, is an equable spiral of Pappus, whose equation is

$$\sin 2\rho = \cos\theta, \quad \text{or} \quad 2\rho = \frac{\pi}{2} - \theta.$$

Another such spiral occurs when the cyclic arcs of the satellite-conic of a spherical quartic are equidistant, by 45°, from its quadruple cyclic arc,
$$1 = (1-\tan^2 x)(1+\tan^2 x+\tan^2 y),$$
or
$$\cos^2\rho = \cos^2\theta, \quad \textit{i.e.}, \quad \rho = n\pi \pm \theta.$$
Such spirals ($\rho = m\theta$) on the sphere are algebraic, since the tangents of the coordinates are periodic functions.*

It has been pointed out by Greatheed (*Math. Journal*, Vol. II., p. 39, 1841) that their rectification can be expressed by an elliptic function of the second order,

$$\int_0^\rho \left(1+\frac{1}{m^2}\sin^2\rho\right)^{\frac{1}{2}} d\rho = -\frac{1}{m}(1+m^2)^{\frac{1}{2}} \int_{\frac{1}{2}\pi}^{\theta} \left(1-\frac{m^2}{m^2+1}\sin^2\theta\right)^{\frac{1}{2}} d\theta$$
$$= \frac{1}{m}(1+m^2)^{\frac{1}{2}} \left\{E_1\left(\frac{m}{\sqrt{1+m^2}}\right) - E\left(\frac{m}{\sqrt{1+m^2}}\right)\right\}.$$

For their quadrature $\displaystyle\iint \sin\rho\, d\theta\, d\rho = \frac{1}{m}(\sin\rho - \rho\cos\rho)$.

The tangent of the polar subtangent arc $\propto (\sin\rho)^2$.

7. To determine the conditions of singularity in the group of spherical quartics

$$\kappa = x(1+x^2+y^2) + \lambda(1+x^2+y^2)^2.$$

In this case $d = 0$, one cyclic arc of the satellite-conic is the (y) arc of reference, and the other coincides with the quadruple cyclic arc of the quartic ($1 = 0$).

The conditions may be found as in § 3,

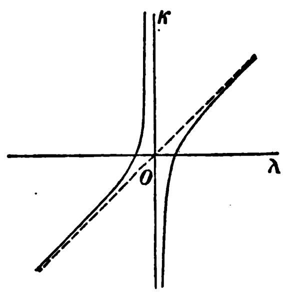


Fig. 7.

* "On y trouve (dans les Collections Mathématiques) la description, sur la sphère, d'une ligne à double courbure remarquable. C'est une spirale que Pappus décrivait, à l'imitation de celle d'Archimède, en faisant mouvoir uniformément un point sur un arc de grand cercle de la sphère, qui tourne lui-même autour de son diamètre (L. 4, Prop. 30). Pappus trouva l'expression de la surface sphérique, comprise entre cette courbe et sa base; premier exemple de la quadrature d'une surface courbe."—Chasles, "Histoire de la Géométrie," p. 29.

or derived from those given in § 3,

$$4\kappa x = x^4 - 1,$$

$$4\lambda x\,(x^2+1)+1+3x^2 = 0.$$

Their eliminant is a sextic in (κ, λ), which determines the mutual relation between κ and λ, when there are singular quartics in the group. This is the first discriminating curve (Fig. 7), and has a centre and two asymptotes, the (κ) axis and the bisector of the angle between the axes.

8. To determine the conditions for a folium-point, or a point of undulation in the same group. By proceeding, as in § 3, or by deducing from the conditions of § 3, one portion of the second discriminating curve is thus obtained,

$$\kappa\,(16\lambda^2+1)^2 = 4\lambda\,(8\kappa\lambda+1)^2 \text{ (Fig. 8).}$$

This quintic has the same centre and asymptotes as the preceding curve of § 4.

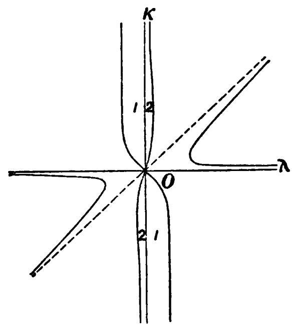


Fig. 8.

A second portion of the discriminating curve is also obtained for the second and fourth quadrants, as in § 4, by the conditions

$$\frac{\kappa u^4}{1+u^2} = -\lambda\,(1+u^2) = \tfrac{1}{2}ud.$$

9. There are no singular quartics in this group, except in the terminal forms, the bifolium quartics correspond to positions of (κ, λ) on either side of the (κ) axis in the first and second quadrants between the serpentine branch of Fig. 7, and the left branch of Fig. 8. A similar limit is similarly defined in the third and fourth quadrants, by symmetry.

10. This group of quartics may take the form

$$\rho_1\frac{b-c}{\sqrt{a}}+\rho_2\frac{c-a}{\sqrt{b}}+\rho_3\frac{a-b}{\sqrt{c}} = 0$$

if $\rho_1^2, \rho_2^2, \rho_3^2$ denote $1+(x+a)^2+y^2$. For, after expansion, the equation becomes

$$\Sigma\,(a^2-2bc)\,(x^2+y^2+1)^2-2abc\,(a+b+c+4x)\,(x^2+y^2+1)+a^2b^2c^2 = 0.$$

The proof is given (*Proceedings*, Vol. XII., No. 167, § 5); but the form in spherics seems barren of results.

11. To determine the foci in quartics of this group.

The tangential equivalent to the equation (A) is

$$S^3-27T^2=0,$$

if the invariants S, T denote the envelopes of arcs $(x\xi+y\eta-1=0)$, which cut the quartic equiharmonically and harmonically,

$$3S=\{2\lambda(1+\xi^2+\eta^2)+\xi^2+\eta^2+d\xi\}^2-\frac{3}{4}(\xi^2+\eta^2)\{(\xi+d)^2+\eta^2(1+d^2)\}$$
$$-3\kappa\lambda(\xi^2+\eta^2)^2,$$

$$27T=\{2\lambda(1+\xi^2+\eta^2)+\xi^2+\eta^2+d\xi\}^3$$
$$-\frac{9}{8}\{2\lambda(1+\xi^2+\eta^2)+\xi^2+\eta^2+d\xi\}(\xi^2+\eta^2)\{(\xi+d)^2+\eta^2(1+d^2)\}$$
$$-9\kappa(\xi^2+\eta^2)^2\left\{\lambda^2(1+\xi^2+\eta^2)+\frac{\lambda}{2}(\xi^2+\eta^2+d\xi)-\frac{3}{16}d^2\eta^2\right\}.$$

If attention be first kept to the terms, which do not contain (κ) in the expansion of $\left(\frac{S}{3}\right)^3-T^2$,

$$\frac{27}{64}(\xi^2+\eta^2)^2(1+\xi^2+\eta^2)\{(\xi+d)^2+\eta^2(1+d^2)\}$$
$$\times\{4\lambda^2(1+\xi^2+\eta^2)+4\lambda(\xi^2+\eta^2+d\xi)-d^2\eta^2\}.$$

Hence $(\xi^2+\eta^2)^2$ measures the tangential equivalent so that it is reduced to the eighth class.

This was anticipated, since the equation (A) denotes, if the coordinates be interpreted as Cartesian, *in plano*, bicircular quartics.

The foci may be found by neglecting $\xi^2+\eta^2+1$, and writing $\xi^2+\eta^2+1-\xi^2-1$ for η^2,

$$3S=(d\xi-1)^2-\frac{3}{4}(d\xi-1)^2-3\kappa\lambda=\frac{1}{4}(d\xi-1)^2-3\kappa\lambda,$$

$$27T=-\frac{1}{8}(d\xi-1)^3-\frac{9\kappa\lambda}{2}(d\xi-1)-\frac{27}{16}\kappa d^2(\xi^2+1).$$

If these values be substituted in the tangential equivalent, there results a quintic, which denotes the collinear foci,

$$\lambda\{(d\xi-1)^2+4\kappa\lambda\}^2+\frac{27}{16}\kappa d^4(\xi^2+1)^2$$
$$+d^2(\xi^2+1)(d\xi-1)\left\{\frac{1}{4}(d\xi-1)^2+9\kappa\lambda\right\}.$$

The origin is a triple focus, since the curve is a class-octavic.*

[The triple focus may be established by reference to the methods of Plane Geometry. (Mr. Esson.)

By writing the equation (A) in the homogeneous form

$$\kappa z^4=z(z+dx)(x^2+y^2+z^2)+\lambda(x^2+y^2+z^2)^2,$$

---

* This gives a quadruple focus, when $d=0$. The tangential equivalent to the resulting concentric circles (see § 3) is

$$\{(2\lambda+1)(\xi^2+\eta)+2\lambda\}^2=(4\kappa\lambda+1)(\xi^2+\eta^2)^2.$$

and for convenience using the language of plane geometry, this is a quartic curve with two flecnodes at the intersections of the line $z=0$ with the curve $x^2+y^2+z^2=0$, or, say, at the points $(z=0,\ x+iy=0)$, $(z=0,\ x-iy=0)$ respectively. At each of these points there is an inflexion touching the conic $(x^2+y^2+z^2=0)$, and an ordinary branch cutting it. Say, the common tangents are T_1 and T_2.

The curve is of the class 8, and it has therefore with the conic 16 common tangents, but among these are included the lines T_1 and T_2 each taken thrice. Say, there are 12 other lines L.

The enumeration of the foci will then be

Disappearing point	(T_1, T_1)	3
" "	(T_2, T_2)	3
point	(T_1, T_2)	9
10 points	(T_1, L)	30
10 points	(T_2, L)	30
$\frac{1}{2}.10.9$ points	(L, L)	45
		120

The total number of foci is $120=\frac{1}{2}.16.15$, as it should be. We have thus a triple focus and five ordinary single foci.]

12. Two foci unite at a node and disappear.

The direct proof would be to find the discriminant of the preceding quintic (§ 11), and identify it with the discriminant of § 3. But the labour may be thus avoided.

The values (1) and (2) of § 3 found for κ and λ in terms of x or $\frac{1}{\xi}$, satisfy the quintic (§ 11) and its first derived function, which is

$$4\lambda d(d\xi-1)\{(d\xi-1)^2+4\kappa\lambda\}+\frac{27}{4}\kappa d^4(\xi^2+1)$$
$$+9\kappa\lambda d^2(3d\xi^2-2\xi+d)+\frac{d^2}{4}(d\xi-1)^2(5d\xi^2+3d-2\xi).$$

The values of κ, λ, $\kappa\lambda$, of § 3 are

$$4\kappa\xi^3=d(1-\xi^4)+2\xi(1+\xi^2),$$
$$4\lambda(1+\xi^2)+2\xi^2+d\xi(\xi^2+3)=0,$$
$$16\kappa\lambda\xi^2+d^2(3-2\xi^2-\xi^4)+8d\xi+4\xi^2=0.$$

If these values of κ, $\kappa\lambda$ be substituted, and then the factor $(d\xi-1)$ be removed, there results

$$\lambda d^2(\xi^2+1)\{(5\xi^2-3)d-8\xi\}$$
$$+\frac{1}{4}d^2\xi\{d^2(5\xi^4+12\xi^2-9)+2d\xi(\xi^2-15)-16\xi^2\}=0,$$

which, after removing a factor, is the preceding value of λ.

This conclusion has been verified for critical cubics and cusped quartics. In the special case, when the cusped quartics unite in a point, $d = -1$, $\kappa = \frac{1}{4}$, $\lambda = 1$, the preceding quintic becomes

$$-\frac{1}{27}(\xi-1)^4(4\xi-3).$$

The former factor denotes the foci, which unite and disappear; the latter, the single focus which remains.

13. This memoir is a companion to a paper on a corresponding group of Bicircular Quartics, which was published in Vol. XII., No. 167, of the *Proceedings.**

If the discriminating curves in the two memoirs be compared, the sequence and variation of quartics will be seen to be similar. The spherical stapete of the first of these curves is represented *in plano* by one cusp at a finite, and another at an infinite distance; the node of the stapete also is withdrawn to infinity.

14. Since duality is perfect in spherics, this memoir exhibits also the enumeration of class-quartics, with a quadruple focus and a triple cyclic arc, if the coordinates x, y be interpreted as the cotangents of the arcs intercepted by a tangent arc to the curve on the arcs of coordinates.†

Sur les Surfaces Parallèles. *By* Prof. A. Mannheim.

[*Read June 9th*, 1881.]

Une droite A, normale en son point a à une surface (S), se déplace en restant constamment normale en ce point à cette surface: ses points ont pour surfaces trajectoires des surfaces parallèles à (S). Les propriétés des trajectoires des points de la normale A, relativement à ces surfaces parallèles, sont des cas particuliers des propriétés générales concernant les trajectoires des points d'une droite mobile quelconque.‡

---

* The case was not considered *in plano*, when d was infinite. The equations of the discriminating curves become

$$256\kappa\lambda^3+27=0, \quad 256\kappa\lambda^3=1.$$

No singular quartic is possible, except in the terminal form of point-quartics.

Moreover, in No. 166, Fig. 6 of the second discriminating curve is incomplete. The omitted portion is found as the locus of (κ, λ) from the equations

$$\kappa u^4 = -\lambda = -\tfrac{1}{2}(u^2+ud).$$

† The paragraphs in [ ] have been added since the paper was read.

‡ Voir: "Sur les trajectoires des points d'une droite mobile dans l'espace" (*Bulletin de la Société Mathématique de France,* Tome 1, page 106), et "Sur les surfaces trajectoires des points d'une figure de forme invariable dont le déplacement est assujetti à quatre conditions" (*Journal de Mathématiques de M. Resal,* 3me Série, Tome 1, page 57).

Je n'examinerai pas, dans le cas des surfaces parallèles, ce que deviennent ces propriétés générales, dont quelques unes sont alors illusoires; je ferai seulement quelques remarques relatives à certaines trajectoires particulières, et je passerai tout de suite à l'objet principal de ce travail, qui est d'établir des propriétés nouvelles dépendant des éléments du 3^e ordre.

§ I.

Soient a, a_1, a_2 ... des points arbitraires marqués sur A. Ces points, que nous appellerons *correspondants*, décrivent pendant le déplacement de A les trajectoires *correspondantes* (a), (a_1), (a_2) ...; nous dirons aussi que les tangentes at, a_1t_1 ... à ces lignes sont des *tangentes correspondantes.*

Pour tous les déplacements de A, à partir d'une position de cette droite, les trajectoires des points correspondants sont normales à A; par suite les surfaces trajectoires de ces points ont, pour normale commune, la droite A. Ceci est vrai, quelle que soit la position de A; donc :

Les normales à une surface (S) *sont aussi normales aux surfaces* (S_1), (S_2), (S_3) *qui lui sont parallèles.*

De là résulte immédiatement que :

(S) *et ses surfaces parallèles ont les mêmes normalies, les mêmes plans de sections principales et la même développée.*

Et par suite, comme on le sait:

Sur les surfaces parallèles, les lignes de courbure sont des lignes correspondantes.

Si (a) est une ligne asymptotique de (S), elle est la ligne de striction de la normalie à (S), qui a cette courbe pour directrice. Il résulte de là que *les lignes correspondantes à une ligne asymptotique ne sont pas des lignes asymptotiques.*

Il faut pourtant excepter le cas où la ligne asymptotique est plane: alors toutes les lignes correspondantes sont des lignes asymptotiques. Par exemple, la cyclide de Dupin est touchée par des plans suivant des circonférences qui sont des lignes asymptotiques et auxquelles correspondent des circonférences analogues sur les surfaces parallèles, qui sont des cyclides comme la première.

Si (a) est une ligne géodésique de (S), ses plans osculateurs sont normaux à cette surface et par suite ils sont tangents à la normalie à (S) dont (a) est la directrice. Cette courbe (a) est alors une ligne asymptotique de cette normalie, et les normales à (S), génératrices de cette normalie, sont les normales principales de (a). Pour qu'une des lignes correspondantes à cette courbe soit aussi une ligne géodésique sur la surface parallèle qui la contient, elle doit donc avoir les mêmes normales principales que (a). Afin qu'il en soit ainsi, en vertu d'une propriété connue, les courbures de (a) doivent être liées par une relation linéaire. Nous voyons donc que :

Parmi les lignes correspondantes à une ligne géodésique (a), on aura une ligne géodésique, si entre les courbures de (a) il existe une relation linéaire.

Circonscrivons des surfaces cylindriques à (S) et aux surfaces qui lui sont parallèles. Les lignes de contact (a), (a_1), (a_2) ... des surfaces (S), (S_1), (S_2) ... avec ces cylindres sont des lignes correspondantes. Elles appartiennent à une normalie (A), commune à toutes ces surfaces, et dont les génératrices sont parallèles à un même plan (V), qui est perpendiculaire aux génératrices des cylindres.

Les traces (a'), (a'_1), (a'_2) ... de ces cylindres sur (V) sont les lignes de contour apparent des surfaces parallèles sur ce plan, et ces lignes sont des courbes parallèles.

La génératrice $a\tau$ du cylindre circonscrit à (S), qui passe par le point a, est la tangente conjuguée de la tangente at à (a). De même en a_1, nous avons les tangentes conjuguées a_1t_1, $a_1\tau_1$, ... ainsi de suite pour a_2, a_3 ... Les droites $a\tau$, $a_1\tau_1$, $a_2\tau_2$ ... sont dans un même plan (C), qui est le plan central de (A) pour la génératrice A.

Le plan de déviation de (S) relatif à la tangente at, c'est-à-dire le lieu des axes de déviation* des sections faites dans (S) par des plans passant par at, est un plan qui contient $a\tau$ et est perpendiculaire à (V). Sa projection sur (V) est alors une simple droite, issue de a'. Cette droite est l'axe de déviation de la courbe de contour apparent (a') pour le point a'. Ceci peut se répéter pour les plans de déviation de (S_1), (S_2) ... relativement aux tangentes correspondantes à at. Mais les axes de déviation des courbes parallèles (a'), (a'_1), (a'_2) ... pour les points correspondants a'_1, a'_2, ..., passent par un même point;† par conséquent *les plans de déviation, dont ces droites sont les projections, passent par une même droite perpendiculaire à (V).*

Appelons e (Fig. 1) le point central de (A), situé sur A, et C la normale en e à la normalie (A), c'est-à-dire la perpendiculaire au plan central (C).

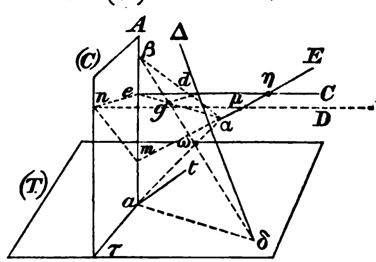


Fig. 1.

La projection de e sur (V) est le centre de courbure de la courbe de contour apparent (a'), et le point de convergence des axes de déviation des courbes de contour apparent est sur la projection de C sur (V). Ce point de convergence

* Il s'agit ici des droites que le Rev. G. Salmon appelle "Axis of aberrancy" (Voir: *Higher Plane Curves*, 2nd Edition, p. 355).

† Cela résulte de la construction de l'axe de déviation en un point d'une courbe, obtenues en faisant usage du centre de courbure de la développée de cette courbe et de cette remarque que des courbes parallèles ont même développée.

est la projection sur ce plan de la droite E par laquelle passent les plans de déviation des surfaces parallèles pour les tangentes correspondantes à at. Cette droite E, parallèle à $a\tau$, rencontre C à angle droit. Nous pouvons maintenant énoncer ce théorème :

TH. I.—*Les plans de déviation de surfaces parallèles, relatifs à des tangentes correspondantes* at, a_1t_1 ... *se coupent suivant une même droite* E. *Cette droite perpendiculaire à la normale commune* A *à ces surfaces, coupe à angle droit la normale* C *à la normalie* (A), *élevée du point central de cette surface situé sur* A.

Menons par at le plan qui coupe (S) suivant une section surosculée par un cercle. Le rayon de ce cercle surosculateur appartient au plan de déviation de (S) relatif à at, il rencontre alors E. Donc :

TH. II.—*Si par des tangentes correspondantes* at, a_1t_1, *on mène des plans qui coupent des surfaces parallèles suivant des sections qui sont respectivement surosculées par un cercle ; les rayons de ces circonférences issues des points correspondants* a, a_1, a_2 ..., *s'appuient sur une même droite* E.

Dans le cas particulier où le plan, qui coupe l'une des surfaces suivant une section surosculée par un cercle contient la normale A, alors la droite E rencontre cette normale et pour que les rayons des cercles surosculateurs des autres sections rencontrent cette droite, ils doivent se confondre avec A.

Nous voyons ainsi que :

TH. III.—*Si parmi les plans menés par des tangentes correspondantes et qui coupent des surfaces parallèles suivant des sections surosculées par un cercle, l'un d'eux est un plan normal à ces surfaces, il en est de même de tous les autres.**

Reprenons le cas où la droite E ne rencontre pas A. Les rayons des cercles surosculateurs s'appuient sur cette droite E et sont tangents au paraboloïde des normales à la normalie (A), relatif à A.

Comme E est parallèle au plan central (C), qui est un plan directeur de ce paraboloïde des normales, ces rayons appartiennent à un paraboloïde qui contient E, et qui est de raccordement avec le paraboloïde des normales, ou encore qui est normal à (A) le long de A. Nous ajoutons alors :

TH. IV. — *Les rayons des cercles surosculateurs, qui entrent dans l'énoncé du théor. II., appartiennent à un paraboloïde qui contient* E *et qui est normal à* (A) *le long de* (A).

---

* Ce théorème, dû à M. Ribaucour, a été l'occasion du présent travail. Il résulte de ce théorème que *les lignes tangentes aux sections normales surosculées par des cercles* (considérés par M. de la Gournerie) *se correspondent sur les surfaces parallèles.* (*Comptes rendus de l'Académie des Sciences*, 15 Mars, 1875.)

Autrement: *Ces rayons rencontrent une infinité de droites parallèles au plan* (C).

Le paraboloïde lieu de ces rayons a pour plan directeur le plan central (C), et, comme C est une de ses génératrices, son second plan directeur est perpendiculaire à (C). Ainsi:

Th. V.—*Les rayons des cercles surosculateurs, qui entrent dans l'énoncé du Th. II., se projettent sur le plan* (C) *suivant des droites parallèles entr'elles.*

Menons par la normale A un plan faisant un angle de 45° avec le plan central (C). Ce plan est normal à la normalie (A) en un point m, qui est à une distance du point central e égale à κ: paramètre de distribution des plans tangents à (A). Ce plan coupe E en un point μ. La droite $m\mu$ est le rayon du cercle surosculateur relatif à la tangente correspondante à at, menée par le point m.

Le plan qui projette orthogonalement $m\mu$, *sur le plan* (C), *est le second plan directeur du paraboloïde des rayons des cercles surosculateurs.*

Ce plan coupe le plan (C, E) suivant une droite D parallèle à C, et il coupe le plan (C) suivant la droite mn. De la construction de mn, il résulte que $en = e\eta$.

Nous connaissons bien maintenant la situation des rayons des cercles surosculateurs et nous allons nous occuper des axes de courbure des sections surosculées par ces cercles.

Construisons l'axe de courbure relatif au point a de la section surosculée par un cercle et qui passe par at. Pour cela, déterminons d'abord le rayon du cercle surosculateur de cette section. Par le point a, je mène un plan parallèle au plan directeur (m, D), il coupe E au point α; $a\alpha$ est le rayon du cercle surosculateur relatif à at. Le plan $(A, a\alpha)$ est normal à la normalie (A) au point a et tangent à cette normalie au point β, qui est tel que $ea \times e\beta = \kappa^2$. Le point β est le centre de courbure de la section faite dans (S) par le plan (A, at),* l'axe de courbure cherché est alors la perpendiculaire $\beta\delta$, abaissée du point β sur le rayon $a\alpha$.

Appelons g le point où cet axe rencontre $e\alpha$. Dans le triangle $a\alpha\beta$, les droites αe, $\beta\delta$ sont deux hauteurs, on a

$$eg \times e\alpha = ea \times e\beta = \kappa^2.$$

D'après cela, on voit que *pour les points correspondants à* a, *les points, tels que* g, *appartiennent à la circonférence transformée de* E *par rayons vecteurs réciproques.*

Le centre de cette circonférence est sur C puisque cette droite est perpendiculaire à E. En élevant alors la perpendiculaire gd à eg, nous obtenons l'extrémité d du diamètre ed de cette circonférence.

* Voir, mon *Cours de Géométrie descriptive*, page 279.

Le plan de l'axe de courbure $\beta\delta$ et de la droite gd contient la normale en β à la normalie (A), il coupe alors le paraboloïde des normales à (A) suivant une autre droite. Mais il coupe déjà la normale C au point δ, donc il rencontre le paraboloïde des normales suivant la génératrice Δ de cette surface qui passe par d. On voit de là que $\beta\delta$ est la projection de Δ sur le plan normal en a à (a). Ainsi :

Th. VI.—*Les sections surosculées par des cercles tangentes en* a_1, a_2, *aux traces de* (A) *sur les surfaces parallèles, ont pour axes de courbure des droites qu'on obtient en projetant une droite* Δ *sur les plans respectivement normaux en ces points à ces traces.*

De là résulte tout de suite que :

Th. VII.—*Le lieu des axes de courbure des sections surosculées par des cercles, qui entrent dans l'énoncé précédent, est un hyperboloïde à une nappe dont les sections circulaires sont respectivement perpendiculaires à* A *et à* Δ.

La circonférence qui a ed pour diamètre est l'une de ces sections, et comme l'on a

$$ed \times e\eta = \kappa^2,$$

nous pouvons dire :

Th. VIII.—*Le produit des distances du point central* e *aux droites* Δ *et* E *est égal au carré du paramètre de distribution des plans tangents à la normalie* (A).

L'axe de courbure $\beta\delta$, qui est perpendiculaire sur $a\alpha$, rencontre ce rayon en un point ω, qui est le centre du cercle surosculateur tangent à at. Pour trouver le lieu des points tels que ω, nous n'avons qu'à chercher la ligne d'intersection du paraboloïde des rayons tels que $a\alpha$ et de l'hyperboloïde lieu des axes tels que $\beta\delta$.

Ces deux surfaces ont en commun la droite A, et la partie restante de leur intersection est alors une cubique gauche. Ainsi :

Th. IX.—*Le lieu des centres des cercles surosculateurs relatifs aux tangentes correspondantes* at, a_1t_1, a_2t_2 … *est une cubique gauche.*

L'axe de courbure $\beta\delta$ rencontre Δ au point δ de la droite $a\delta$ normale à (A). Ce point δ, qui est dans le plan tangent en a à (S), est alors le centre de courbure géodésique de la section surosculée par un cercle dont le plan est ωat. Comme le point a est arbitraire nous voyons que :

Th. X.—*Les courbes surosculées par des cercles, tangentes aux traces d'une normalie sur des surfaces parallèles entr' elles, ont leurs centres de courbure géodésique sur une même droite.*

La droite Δ, génératrice du paraboloïde des normales à (A), se projette sur le plan tangent (T) suivant une parallèle à la trace du plan

central (C) sur le même plan (T). Le segment $a_1\delta_1$* analogue à $a\delta$ étant parallèle à (T) se projette sur ce plan sans altération de grandeur. Au moyen de cette projection on voit que ces segments sont inversement proportionnels aux cosinus des angles qu'ils font avec le plan qui touche (A) au point à l'infini sur A.

La droite A, et la génératrice de (A) qui lui est infiniment voisine, déterminent sur (a) et sur la ligne correspondante (a_1) des arcs infiniment petits qui sont dans le même rapport. De là ce théorème:

Th. XI.—*Les sections surosculées par des cercles, tangentes en* a_1, a_2 ... *aux traces d'une normalie sur des surfaces parallèles, ont, en ces points correspondants, des angles de contingence géodésique égaux entr'eux.*

§ II.

Les théorèmes que nous venons de démontrer ne sont pas particuliers aux sections surosculées par des cercles tangentes aux traces d'une normalie sur des surfaces parallèles. Nous allons trouver des théorèmes analogues pour d'autres courbes en suivant une route absolument inverse de celle suivie jusqu'à présent.

Commençons par déterminer les centres de courbure géodésique de ces courbes. Pour cela, j'ai besoin de rappeler la construction de certaines droites dont j'ai eu déjà l'occasion de faire voir l'utilité pour démontrer des propriétés qui dépendent des éléments du 3^e ordre.†

Soient b et c (Fig. 2) les centres de courbure principaux des surfaces (S), (S_1) ... situés sur la normale A. Toutes les normalies, dont les directrices sont tangentes en a à (a) sont osculatrices entr'elles aux points b et c. En ces points, les indicatrices de ces normalies ont les mêmes asymptotes : l'une de ces droites est A, et les autres sont bb', cc'. Ce sont ces deux droites dont je vais faire usage.

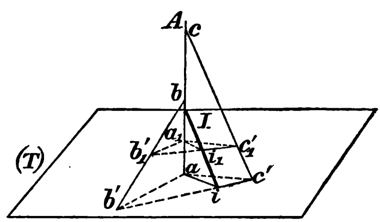


Fig. 2.

Les courbes dont je parlerai d'abord et dont voici la définition sont les *courbes à courbure normale constante.*‡ Appelons les Σ. La courbe Σ est telle que les plans normaux à (S), qui lui sont tangents, déterminent dans cette surface des sections pour lesquelles les rayons de courbure sont égaux ; ces rayons de courbure étant relatifs aux points où les plans normaux à (S) touchent Σ.

Traçons sur (S) une courbe Σ tangente en a à at. Pour déterminer

---

* Ce segment n'est pas représenté sur la figure.

† *Comptes rendus des séances de l'Academie des Sciences.* Séances du 22 Mars, 1875, et du 6 Mars, 1876.

‡ Voir, Ribaucour: *Comptes rendus de l'Académie des Sciences,* Séance du 15 Mars, 1875.

le centre de courbure géodésique de Σ au point a, on a la construction suivante :*

On prend sur le plan tangent (T) les traces b', c' des droites bb', cc' ; on joint les points b' et c' par une droite; cette droite coupe en i la normale ai à (a), le point i est le centre de courbure géodésique de Σ.

Traçons sur (S_1) une courbe Σ tangente en a_1 à a_1t_1, et construisons, au moyen d'une droite $b'_1c'_1$, le centre de courbure géodésique i_1 de cette courbe.

Pour toutes les surfaces parallèles à (S) nous aurons de la même manière des points tels que i, i_1 ... *Je dis que tous ces points sont en ligne droite.*

Les droites ai, a_1i_1 sont normales à la normalie (A) et appartiennent alors au paraboloïde des normales à cette normalie. Ce paraboloïde a pour plan directeur le plan (T) et le plan central (C).

Les droites $b'c'$, $b'_1c'_1$ ... appartiennent aussi à un paraboloïde; celui-ci a pour plan directeur le plan (T) et pour directrices les droites bb', cc'.

Ces deux paraboloïdes, qui ont un plan directeur commun, ont aussi en commun les normales en b et c aux nappes de la développée de (S).

La partie restante de leur intersection est donc une simple droite. Nous l'appellerons I. On a ce théorème :

TH. XII.—*Les centres de courbure géodésique des courbes à courbure normale constante, tangentes aux traces d'une normalie sur des surfaces parallèles entr'elles, sont en ligne droite.*

Dans le cas particulier où l'un des centres de courbure géodésique est à l'infini, la droite I est toute entière à l'infini et nous avons comme conséquence un théorème analogue au théorème III.

Comme précédemment on démontre que :

TH. XIII.—*Les angles de contingence géodésique des courbes à courbure normale constante, qui sont tangentes aux traces d'une normalie sur des surfaces parallèles, sont égaux entr'eux.*

Le plan (A, ai), (Fig. 3), qui est normal en a à Σ, est tangent à la normalie (A) au point β, centre de courbure de la section faite dans (S) par le plan (A, at). La droite βi est alors l'axe de courbure de Σ relatif au point a. Mais puisque I est une génératrice du paraboloïde des normales à (A) cette droite est rencontrée non seulement par ai, mais aussi par la normale au point β à cette

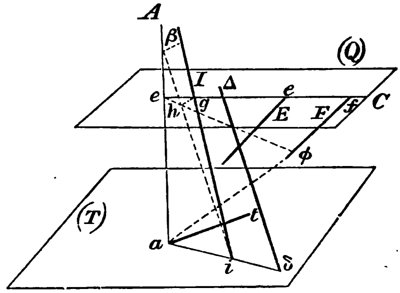


FIG. 3.

* Voir, Ribaucour: *loc. cit.*

surface, c'est à dire par la perpendiculaire élevée du point β au plan (Aai). On voit ainsi que βi est la projection de I.* Ceci est vrai pour les courbes Σ relatives aux lignes correspondantes (a), (a_1) ... Donc:

TH. XIV.—*Le lieu des axes de courbure des courbes à courbure normale constante, tangentes aux traces d'une normalie sur des surfaces parallèles, est un hyperboloïde dont les plans des sections circulaires sont respectivement perpendiculaires à A et I.*

La normale principale de Σ au point a est la perpendiculaire abaissée de ce point sur βi. Appelons ϕ le point où cette perpendiculaire rencontre le plan (Q) perpendiculaire à A au point central e, et appelons h le point où βi rencontre le même plan. Les points e, h, ϕ sont en ligne droite, et l'on a $eh \times e\phi = ea \times e\beta = \kappa^2$;

mais les points tels que h appartiennent à la section circulaire suivant laquelle le plan (Q) coupe l'hyperboloïde des axes de courbure des courbes Σ: donc le lieu des points tels que ϕ est une droite, transformée par rayons vecteurs réciproques de cette circonférence. Cette droite que nous appellerons F est perpendiculaire au diamètre eg de la circonférence transformée. Elle est donc parallèle à la droite E, dont nous avons parlé précédemment.

Les normales principales des courbes Σ relatives aux points correspondants a, a_1, a_2, sont donc des droites qui s'appuient sur une droite F et qui sont tangentes au paraboloïde des normales à A. Comme F est parallèle au plan directeur de ce paraboloïde, nous pouvons dire:

TH. XV.—*Pour les points correspondants a, a_1, a_2, les normales principales des courbes à courbure normale constante, menées tangentiellement aux traces d'une normalie sur les surfaces parallèles, appartiennent à un paraboloïde hyperbolique.*

Les plans directeurs de ce paraboloïde sont le plan central (C) et un plan perpendiculaire à I. Rapprochons ce que nous venons de trouver de ce que nous avons établi précédemment pour les sections surosculées par des cercles.

En vertu d'un beau théorème dû à M. Ribaucour,† on sait que:

$$ai = \frac{2}{3}\,a\delta.$$

On en conclut facilement que:

$$\text{Tang}\,(A, I) = \frac{2}{3}\,\text{Tang}\,(A, \Delta).$$

* On peut remarquer que la projection de I sur le plan (A, C) est parallèle à A et que cette projection contient le point de rencontre des projections sur le même plan des droites bb', cc' représentées (Fig. 2).

† *Loc. cit.*

Comme les droites A, I, Δ sont parallèles au plan (C), on a :

$$\frac{ej}{ed} = \frac{ai}{a\delta} = \frac{2}{3}.$$

Mais
$$ej = \frac{\kappa^2}{ef}, \quad ed = \frac{\kappa^2}{e\eta},$$

on a donc aussi :
$$\frac{e\eta}{ef} = \frac{2}{3}.$$

Telle est la relation de position des droites E, F.

On sait que $3e\eta$ est égal au rayon de courbure de la développée des courbes de contour apparent (a'), (a'_1) ... des surfaces parallèles ; nous voyons donc que *ce rayon de courbure est aussi égal à $2ef$.**

§ III.

On voit, par ce que nous venons de dire des courbes Σ, qu'il nous a suffi de démontrer le Th. XII., pour arriver à tous les théorèmes analogues à ceux que nous avions trouvés précédemment dans le § 1er.

On a encore des théorèmes analogues à ceux-ci pour les lignes dont je vais parler maintenant.

Traçons sur (S) et tangentiellement à (a) en a une courbe qui coupe sous des angles égaux les lignes de courbure de l'un des systèmes de (S). Nous appellerons cette courbe *une ligne trajectoire des lignes de courbure de* (S). Nous aurons de même sur (S_1), tangentiellement à (a_1) en a, une ligne trajectoire des lignes de courbure de (S_1), et ainsi de suite pour les surfaces parallèles à (S).

Le centre de courbure géodésique de la ligne trajectoire tracée sur (S) tangentiellement à (a) en a, s'obtient en prenant sur le plan (T) le point de rencontre l de la normale al à cette courbe avec la trace du paraboloïde des huit droites relatif à (S).† Ce paraboloïde est le même pour toutes les surfaces parallèles à (S), et les droites telles que al appartiennent au paraboloïde des normales à la normalie (A).

Ces deux paraboloïdes ont en commun : les normales élevées des centres de courbure principaux b et c aux nappes de la développée de (S), et comme ils ont un plan directeur commun la partie restante de leur ligne d'intersection est une droite. On a donc ce théorème :

Th. XVI.—*Les lignes trajectoires des lignes de courbure des surfaces parallèles, qui sont menées tangentiellement aux traces d'une normalie sur ces surfaces, ont leur centre de courbure géodésique sur une même droite.*

De là, on peut déduire, comme au § II., toute une suite de théorèmes.

---

* Ce rayon de courbure est aussi égal à $\frac{2\kappa^2}{ej}$. Je montrerai, dans une autre occasion, comment on peut établir directement cette relation.

† *Comptes rendus des Séances de l'Académie des Sciences.* Séance du 2 Avril, 1877.

On the Gaussian Theory of Surfaces. *By* Prof. Cayley.

[*Read June 9th*, 1881.]

In the Memoir, Bour, "Théorie de la déformation des Surfaces" (*Jour. de l'Ec. Polyt.*, Cah. 39, 1862, pp. 1—148), the author working with the form $ds^2 = dv^2 + g^2du^2$ (as a special case of Gauss's formula $ds^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2$) obtains, p. 29, the following equations which he calls *fundamental*:—

$$\text{[IV.]} \ldots\ldots \begin{cases} \dfrac{1}{g}\dfrac{dg_1}{dv} = T^2 - HH_1, \\ \dfrac{dT}{du} + \dfrac{d\,.\,Hg}{dv} - H_1 g_1 = 0, \\ \dfrac{d\,.\,Tg^2}{dv} + g\dfrac{dH_1}{du} \qquad = 0, \end{cases}$$

where g_1 is written to denote $\dfrac{dg}{dv}$, and where (see p. 26)

H is the curvature of the normal section containing the tangent to the curve $v = \text{const.}$,

H_1 is the curvature of the normal section at right angles to the preceding, containing the tangent to the (geodesic) curve $u = \text{constant}$,

T is the torsion of the same geodesic curve;

or, what is the same thing, see p. 25, the quadric equation for the determination of the principal radii of curvature at the point of the surface is

$$\left(\frac{1}{\rho} - H\right)\left(\frac{1}{\rho} - H_1\right) - T^2 = 0.$$

Writing for greater convenience K in place of the suffixed letter H_1, also V instead of g (so that the differential formula is $ds^2 = dv^2 + V^2du^2$),

$$\text{the equations become} \begin{cases} \dfrac{1}{V}\dfrac{d^2V}{dv^2} = T^2 - HK, \\ \dfrac{dT}{du} + \dfrac{d\,.\,HV}{dv} - K\dfrac{dV}{dv} = 0, \\ \dfrac{d\,.\,TV^2}{dv} + V\dfrac{dK}{du} \qquad = 0; \end{cases}$$

or, if we use the suffix 1 to denote differentiation in regard to v, and the suffix 2 to denote differentiation in regard to u, then the equations

are
$$\frac{V_{11}}{V} = T^2 - HK,$$
$$T_2 + (HV)_1 - KV_1 = 0,$$
$$(TV^2)_1 + K_2 V = 0,$$

or, what is the same thing,

$$\begin{cases} V_{11} = V(T^2 - HK), \\ T_2 + H_1 V + (H-K) V_1 = 0, \\ T_1 V + 2TV_1 + K_2 \qquad = 0. \end{cases}$$

I wish to show how these formulæ connect themselves with formulæ belonging to the general form $ds^2 = Edp^2 + 2Fdp\,dq + Gdq^2$. These involve not only Gauss' coefficients E, F, G, but also the coefficients E', F', G' belonging to the inflexional tangents; and, for convenience, I quote the system of definitions, Salmon's "Geometry of Three Dimensions," 3rd ed., 1874, p. 251, viz.,

$$dx,\ dy,\ dz = a\,dp + a'dq,\ \ b\,dp + b'dq,\ \ c\,dp + c'dq;$$
$$d^2x = \alpha\,dp^2 + 2\alpha' dp\,dq + \alpha'' dq^2,$$
$$d^2y = \beta\,dp^2 + 2\beta' dp\,dq + \beta'' dq^2,$$
$$d^2z = \gamma\,dp^2 + 2\gamma' dp\,dq + \gamma'' dq^2;$$
$$A,\ B,\ C = bc' - b'c,\ \ ca' - c'a,\ \ ab' - a'b; \quad V^2 = EG - F^2;$$
$$E' = A\alpha + B\beta + C\gamma, \quad F' = A\alpha' + B\beta' + C\gamma', \quad G' = A\alpha'' + B\beta'' + C\gamma'',$$

so that E', F', G' are, in fact, the determinants

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ \alpha, & \beta, & \gamma \end{vmatrix}, \quad \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ \alpha', & \beta', & \gamma' \end{vmatrix}, \quad \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}.$$

The equation for the determination of the principal radii of curvature is
$$(E'\rho - EV)(G'\rho - GV) - (F'\rho - FV)^2 = 0,$$
which, in the particular case $F = 0$ (and therefore $V^2 = EG$), becomes
$$(E'\rho - EV)(G'\rho - GV) - F'^2 . \rho^2 = 0,$$
or, as this may be written,
$$\left(\frac{1}{\rho} - \frac{E'}{EV}\right)\left(\frac{1}{\rho} - \frac{G'}{GV}\right) - \frac{F'^2}{EGV^2} = 0,$$
an equation which corresponds with Bour's form
$$\left(\frac{1}{\rho} - K\right)\left(\frac{1}{\rho} - H\right) - T^2 = 0,$$

and becomes identical with it, if

$$E' = EVK, \quad G' = GVH, \quad F' = -V^2T.$$

But, making p, q correspond to Bour's variables, p to v, and q to u, it is necessary to show that the foregoing values (and not the interchanged values $E' = GVH$, $G' = EVK$) are the correct ones. We have, Salmon, p. 254,

$$\left\| \begin{matrix} dq, & \rho E' - VE, & \rho F' - VF \\ -dp, & \rho F' - VF, & \rho G' - VG \end{matrix} \right\| = 0\,;$$

or, putting herein $F = 0$, the equations may be written

$$\frac{dq}{-dp} = \frac{E'}{F'}\left(1 - \frac{VE}{\rho E'}\right) = \frac{F'}{G'} \div \left(1 - \frac{VG}{\rho G'}\right);$$

or, we see that to $dq = 0$ corresponds the value $\frac{1}{\rho} = \frac{E'}{EV}$, and to $dp = 0$ the value $\frac{1}{\rho} = \frac{G'}{GV}$. Hence the former of these values of $\frac{1}{\rho}$ corresponds to Bour's $du = 0$, that is, to his $\frac{1}{\rho} = K$; and the latter to Bour's $dv = 0$, that is, to his $\frac{1}{\rho} = H$; or, the values are as stated,

$$E' = EVK, \quad G' = GVH.$$

The formula $ds^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2$ agrees with Bour's $ds^2 = dv^2 + g^2 du^2$, if $p = u$, $q = v$, $E = 1$, $F = 0$, $G = g^2$. With these values, $V^2 = EG - F^2 = g^2$, or say $g = V$, and Bour's equation is, as it was before written, $ds^2 = dv^2 + V^2 du^2$. And we have to find the three equations which, putting therein $p = u$, $q = v$, $E = 1$, $F = 0$, $G = V^2$, $E' = VK$, $F' = -V^2T$, $G' = V^3H$, reduce themselves to Bour's equations.

The first of these is nothing else than the equation for the measure of curvature, viz., Salmon, p. 262 (but, using the suffixes 1 and 2 to denote differentiation in regard to p and q respectively), this is

$$\begin{aligned} 4(E'G' - F'^2) = \;& E(E_2 G_2 - 2F_1 G_2 + G_1^2) \\ & + F(E_1 G_2 - E_2 G_1 - 2E_2 F_1 + 4F_1 F_2 - 2F_1 G_1) \\ & + G(E_1 G_1 - 2E_1 F_2 + E_2^2) \\ & - 2(EG - F^2)(E_{22} - 2F_{12} + G_{11}). \end{aligned}$$

In fact, writing herein $E = 1$, $F = 0$, and therefore the differential coefficients of E and F each $= 0$, the equation becomes

$$4(E'G' - F'^2) = G_1^2 - 2GG_{11},$$

which is

$$4V^4(HK-T^2) = (2VV_1)^2 - 2V^2(2V_1^2 + 2VV_{11}), = -4V^3V_{11};$$

or finally it is

$$V_{11} = V(T^2 - HK).$$

The other two of Bour's equations are derived from equations which give respectively the values of $E_2' - F_1'$ and $F_2' - G_1'$; viz., starting from the equations

$$\begin{aligned} E' &= A\alpha + B\beta + C\gamma, \\ F' &= A\alpha' + B\beta' + C\gamma', \\ G' &= A\alpha'' + B\beta'' + C\gamma'', \end{aligned}$$

we see at once that E_2' and F_1' contain, E_2' the terms $A\alpha_2 + B\beta_2 + C\gamma_2$, and F_1' the terms $A\alpha_1' + B\beta_1' + C\gamma_1'$, which are equal to each other ($\alpha_2 = \alpha_1'$ since α and α' are the differential coefficients x_{11}, x_{12} of x, and so $\beta_2 = \beta_1'$ and $\gamma_2 = \gamma_1'$). Hence

$$E_2' - F_1' = A_2\alpha + B_2\beta + C_2\gamma - A_1\alpha' - B_1\beta' - C_1\gamma';$$

and similarly

$$F_2' - G_1' = A_2\alpha' + B_2\beta' + C_2\gamma' - A_1\alpha'' - B_1\beta'' - C_1\gamma''.$$

Here, from the values of A, B, C, we have

$$\begin{aligned} A &= bc' - cb'; & A_1 &= \beta c' - \gamma b' + b\gamma' - c\beta'; & A_2 &= \beta'c' - \gamma'b' + b\gamma'' - c\beta''; \\ B &= ca' - ac'; & B_1 &= \gamma a' - \alpha c' + c\alpha' - a\gamma'; & B_2 &= \gamma'a' - \alpha'c' + c\alpha'' - a\gamma''; \\ C &= ab' - ba'; & C_1 &= \alpha b' - \beta a' + a\beta' - b\alpha'; & C_2 &= \alpha'b' - \beta'a' + a\beta'' - b\alpha''; \end{aligned}$$

and, substituting, we find

$$\begin{aligned} E_2' - F_1' &= 2a'\alpha\alpha' + a\alpha''\alpha, \\ F_2' - G_1' &= -2a\alpha'\alpha'' - a'\alpha''\alpha, \end{aligned}$$

if, for shortness, $a'\alpha\alpha'$ denotes the determinant

$$\begin{vmatrix} a', & \alpha, & \alpha' \\ b', & \beta, & \beta' \\ c', & \gamma, & \gamma' \end{vmatrix},$$

and so for the other like symbols. Observe that with

$$\begin{vmatrix} a, & a', & \alpha, & \alpha', & \alpha'' \\ b, & b', & \beta, & \beta', & \beta'' \\ c, & c', & \gamma, & \gamma', & \gamma'' \end{vmatrix}$$

we have in all 10 determinants, viz., these are $aa'\alpha, = E'$; $aa'\alpha', = F'$; $aa'\alpha'', = G'$; $\alpha\alpha'\alpha''$; and the six determinants $a\alpha\alpha'$, $a\alpha'\alpha''$, $a\alpha''\alpha$; $a'\alpha\alpha'$, $a'\alpha'\alpha''$, $a'\alpha''\alpha$. The foregoing expressions of $E_2' - F_1'$ and $F_2' - G_1'$ respectively, substituting therein for the determinants $a'\alpha\alpha'$, $a\alpha''\alpha$, $a\alpha'\alpha''$, $a'\alpha''\alpha$ their values as about to be obtained, are the required two equations.

We have

$$\begin{array}{ll} aa\ +bb\ +cc\ = E, & aa'\ +bb'\ +cc'\ = F, \\ a'a+b'b\ +c'c\ = F, & a'a'+b'b'\ +c'c'\ = G, \\ \alpha a\ +\beta b\ +\gamma c\ = \tfrac{1}{2}E_1, & \alpha a'\ +\beta b'\ +\gamma c'\ = F_1-\tfrac{1}{2}E_2, \\ \alpha' a+\beta' b+\gamma' c\ = \tfrac{1}{2}E_2, & \alpha' a'+\beta' b'\ +\gamma' c' = \tfrac{1}{2}G_1, \\ \alpha'' a+\beta'' b+\gamma'' c = F_2-\tfrac{1}{2}G_1, & \alpha'' a'+\beta'' b'+\gamma'' c' = \tfrac{1}{2}G_2\,; \end{array}$$

and if from the first five equations, regarded as equations linear in (a, b, c), we eliminate these quantities, and from the second five equations, regarded as linear in (a', b', c'), we eliminate these quantities, we obtain two sets each of five equations,

$$\left\Vert \begin{array}{lllll} a, & a', & \alpha, & \alpha', & \alpha'' \\ b, & b', & \beta, & \beta', & \beta'' \\ c, & c', & \gamma, & \gamma', & \gamma'' \\ E, & F, & \tfrac{1}{2}E_1, & \tfrac{1}{2}E_2, & F_2-\tfrac{1}{2}G_1 \end{array} \right\Vert = 0, \text{ and } \left\Vert \begin{array}{lllll} a, & a', & \alpha, & \alpha', & \alpha'' \\ b, & b', & \beta, & \beta', & \beta'' \\ c, & c', & \gamma, & \gamma', & \gamma'' \\ F, & G, & F_1-\tfrac{1}{2}E_2, & \tfrac{1}{2}G_1, & \tfrac{1}{2}G_2 \end{array} \right\Vert .$$

Or, as these may be written,

$$\begin{array}{l} F\,\alpha\alpha'\alpha'' -\tfrac{1}{2}E_1 a'\alpha'\alpha'' -\tfrac{1}{2}E_2\, a'\alpha''\alpha - (F_2-\tfrac{1}{2}G_1)\, a'\alpha\alpha' = 0, \\ -E\,\alpha\alpha'\alpha'' +\tfrac{1}{2}E_1 a\alpha'\alpha'' +\tfrac{1}{2}E_2\, a\alpha''\alpha + (F_2-\tfrac{1}{2}G_1)\, a\alpha\alpha' = 0\,; \end{array}$$

$$\begin{array}{llll} E\, a'\alpha'\alpha'' - & F\ a\alpha'\alpha'' +\tfrac{1}{2}E_2\, G' - (F_2-\tfrac{1}{2}G_1)\, F' & & = 0, \\ E\, a'\alpha''\alpha - & F\ a\alpha''\alpha -\tfrac{1}{2}E_1\, G' + (F_2-\tfrac{1}{2}G_1)\, E' & & = 0, \\ E\, a'\alpha\alpha' - & F\ a\alpha\alpha' +\tfrac{1}{2}E_1\, F' - & \tfrac{1}{2}E_2\, E' & = 0\,; \end{array}$$

and

$$\begin{array}{llll} G\,\alpha\alpha'\alpha'' - (F_1-\tfrac{1}{2}E_2)\, a'\alpha'\alpha'' - \tfrac{1}{2}G_1\, a'\alpha''\alpha & & -\tfrac{1}{2}G_2\, a'\alpha\alpha' & = 0, \\ -F\,\alpha\alpha'\alpha'' + (F_1-\tfrac{1}{2}E_2)\, a\alpha'\alpha'' + \tfrac{1}{2}G_1\, a\alpha''\alpha & & +\tfrac{1}{2}G_2\, a\alpha\alpha' & = 0, \\ F\, a'\alpha'\alpha'' & -\ G\ \ a\alpha'\alpha'' + \tfrac{1}{2}G_1\, G' & -\tfrac{1}{2}G_2 F' & = 0, \\ F\, a'\alpha''\alpha & -\ G\ \ a\alpha''\alpha - (F_1-\tfrac{1}{2}E_2)\, G' & +\tfrac{1}{2}G_2 E' & = 0, \\ F\, a'\alpha\alpha' & -\ G\ \ a\alpha\alpha' + (F_1-\tfrac{1}{2}E_2)\, F' & -\tfrac{1}{2}G_1 E' & = 0. \end{array}$$

Attending in each set only to the third, fourth, and fifth equations, and combining these in pairs, we obtain

$$\begin{array}{l} V^2\, a\alpha'\alpha'' + (\ \tfrac{1}{2}FG_1 - FF_2 + \tfrac{1}{2}EG_2)\ F' + (-\tfrac{1}{2}EG_1 + \tfrac{1}{2}FE_2)\qquad G' = 0, \\ V^2\, a'\alpha'\alpha'' + (\ \tfrac{1}{2}GG_1 - GF_2 + \tfrac{1}{2}FG_2)\ F' + (-\tfrac{1}{2}FG_1 + \tfrac{1}{2}GE_2)\qquad G' = 0\,; \end{array}$$

$$\begin{array}{l} V^2\, a\alpha''\alpha + (-\tfrac{1}{2}FE_1 + EF_1 - \tfrac{1}{2}EE_2)\ G' + (-\tfrac{1}{2}FG_1 + FF_2 - \tfrac{1}{2}EG_2)\ E' = 0, \\ V^2\, a'\alpha''\alpha + (-\tfrac{1}{2}GE_1 + FF_1 - \tfrac{1}{2}FE_2)\ G' + (-\tfrac{1}{2}GG_1 + GF_2 - \tfrac{1}{2}FG_2)\ E' = 0\,; \end{array}$$

$$\begin{array}{l} V^2\, a\alpha\alpha' + (\ \tfrac{1}{2}EG_1 - \tfrac{1}{2}FE_2)\qquad E' + (\ \tfrac{1}{2}FE_1 - EF_1 + \tfrac{1}{2}EE_2)F' = 0, \\ V^2\, a'\alpha\alpha' + (\ \tfrac{1}{2}FG_1 - \tfrac{1}{2}GE_2)\qquad E' + (\ \tfrac{1}{2}GE_1 - FF_1 + \tfrac{1}{2}FE_2)F' = 0. \end{array}$$

We thus obtain

$$E_2' - F_1 = \frac{2}{V^2}\{(\qquad -\tfrac{1}{2}FG_1 + \tfrac{1}{2}GE_2)E' + (-\tfrac{1}{2}GE_1 + FF_1 - \tfrac{1}{2}FF_2)F'\}$$

$$+\frac{1}{V^2}\{(\ \tfrac{1}{2}FE_1 - EF_1 + \tfrac{1}{2}EE_2)G' + (\ \tfrac{1}{2}FG_1 - FF_2 + \tfrac{1}{2}EG_2)E'\},$$

$$F_2' - G_1' = \frac{2}{V^2}\{(\ \tfrac{1}{2}FG_1 - FF_2 + \tfrac{1}{2}EG_2)F' + (\qquad -\tfrac{1}{2}EG_1 + \tfrac{1}{2}FE_2)G'\}$$

$$+\frac{1}{V^2}\{(-\tfrac{1}{2}GE_1 + FF_1 - \tfrac{1}{2}FE_2)G' + (-\tfrac{1}{2}GG_1 + GF_2 - \tfrac{1}{2}FG_2)E'\};$$

or finally

$$E_2' - F_1' = \frac{1}{V^2}\{(-\tfrac{1}{2}FG_1 + GE_2 - FF_2 + \tfrac{1}{2}EG_2)E' \\ + (-GE_1 + 2FF_1 - FF_2)F' + (\tfrac{1}{2}FE_1 - EF_1 + \tfrac{1}{2}EE_2)G'\},$$

$$F_2' - G_1' = \frac{1}{V^2}\{(-\tfrac{1}{2}GG_1 + GF_2 - \tfrac{1}{2}FG_2)E' \\ + (FG_1 - 2FF_2 + EG_2)F' + (-\tfrac{1}{2}GE_1 + FF_1 - EG_1 + \tfrac{1}{2}FE_2)G'\},$$

which are the required formulæ; and which may, I think, be regarded as new formulæ in the Gaussian theory of surfaces.

Writing herein as before, the first of these becomes

$$(VK)_2 + (V^2T)_1 = \frac{1}{V^2}\{\tfrac{1}{2}(V^2)_2\,VK\}, = V_2K,$$

that is, $$V_2K + VK_2 + V^2T_1 + 2VV_1T = V_2K;$$

or finally $$VT_1 + 2TV_1 + K_2 = 0,$$

which is Bour's third equation. And the second equation becomes

$$-(V^2T)_2 - (V^3H)_1 = \frac{1}{V^2}\{-\tfrac{1}{2}V^2(V^2)_1VK + (V^2)_2(-V^2T) - (V^2)_1V^3H\},$$

$$= -V^2V_1K - 2VV_2T - 2V^3V_1H,$$

that is,

$$-V^2T_2 - 2VV_2T - V^3H_1 - 3V^2V_1H = -V^2V_1K - 2VV_2T - 2V^3V_1H;$$

or finally $$T_2 + VH_1 + (H-K)V_1 = 0,$$

which is Bour's second equation.

Theorems in the Calculus of Operations. *Part 2.*

By J. J. WALKER.

[*Read June 9th,* 1881.]

The development of $D^n uv\phi^n$ given in the first Part (*Proceedings*, Vol. xi., p. 110) of this paper, viz., u, v, ϕ being any three functions of x, and v' being Dv, or $\frac{dv}{dx}$,

$$D^n uv\phi^n = vD^n u\phi^n + nv'\phi D^{n-1}u\phi^{n-1} + \frac{n\,.\,n-1}{1\,.\,2}Dv'\phi^2\,.\,D^{n-2}u\phi^{n-2} + \ldots + D^{n-1}v'\phi^n\,.\,u \;\ldots\ldots (g),$$

admits of an obvious generalization. If D' stands for the operator $D\phi$, so that $D'^r u = D^r u\phi^r$ [*not* $(D\phi)^r u$]; then, $f(x)$ being any rational algebraic function of positive powers of x,

$$f(D')\,uv = vf(D')\,u + v'\phi f'(D')\,u + \frac{1}{1\,.\,2}D'v'\phi\,.\,f''(D')\,u + \ldots + D'^{n-1}v'\phi^{n-1}\,.\,u \ldots\ldots\ldots (h),$$

which, when ϕ is a constant, gives Hargreave's Series I., *Phil. Trans.*, 1848.

The development (g) is not explicitly symmetrical in u, v; but it gives an explicitly symmetrical relation, of some interest, as follows:—

Observing that $\qquad v'\phi = Dv\phi - v\phi'$,

$$Dv'\phi^2 = D^2v\phi^2 - 2Dv\phi\phi' \;\ldots\; D^{r-1}v'\phi^r = D^r v\phi^r - rD^{r-1}v\phi^{r-1}\phi';$$

$$D^n uv\phi^n = vD^n u\phi^n + nDv\phi . D^{n-1}u\phi^{n-1} + \frac{n.n-1}{1\,.\,2}D^2v\phi^2.D^{n-2}u\phi^{n-2} + \ldots + D^n v\phi^n.u$$
$$-n\,[v\phi' D^{n-1}u\phi^{n-1} + (n-1)\,Dv\phi\phi'\,.\,D^{n-2}u\phi^{n-2} + \ldots$$
$$\ldots + (n-1)\,D^{n-2}v\phi^{n-2}\phi'\,.\,Du\phi + D^{n-1}v\phi^{n-1}\phi'\,.\,u].$$

Now, the part within brackets is what the other part of this development becomes, on changing v into $v\phi'$ and n into $n-1$; so that, writing $F(v, n)$ for that former part,

$$\begin{aligned}
D^n uv\phi^n &= F(v, n) - nF(v\phi', n-1),\\
D^{n-1}uv\phi^{n-1}\phi' &= F(v\phi', n-1) - (n-1)\,F(v\phi'^2, n-2),\\
D^{n-2}uv\phi^{n-2}\phi'^2 &= F(v\phi'^2, n-2) - (n-2)\,F(v\phi'^3, n-3),\\
\ldots \quad \ldots &\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\
Duv\phi\phi'^{n-1} &= F(v\phi'^{n-1}, 1) - F(v\phi'^n, 0)\\
&= v\phi'^{n-1}Du\phi + Dv\phi\phi'^{n-1}.u - uv\phi'^n.
\end{aligned}$$

Multiplying these equalities by 1, n, $n(n-1) \ldots \underline{|n}$ respectively, and adding up,

$$D^n uv\phi^n + nD^{n-1}uv\phi^{n-1}\phi' + n(n-1)D^{n-2}uv\phi^{n-2}\phi'^2 + \ldots + \underline{|n}\, uv\phi'^n$$

$$= vD^n u\phi^n + nDv\phi \,.\, D^{n-1}u\phi^{n-1} + \frac{n(n-1)}{1\,.\,2} D^2 v\phi^2 \,.\, D^{n-2}u\phi^{n-2} + \ldots + D^n v\phi^n \,.\, u \qquad \ldots\ldots (i).$$

And this, using the same notation as before, viz., $D'^r u = D^r u\phi^r$, admits of the generalization,

$$f(D')\, uv + f'(D')\, uv\phi' + f''(D')\, uv\phi'^2 + \ldots$$

$$= vf(D')\, u + D'v \,.\, f'(D')\, u + \frac{1}{1\,.\,2} D'^2 v \,.\, f''(D')\, u + \ldots \quad \ldots (j).$$

In the publication of the theorem which follows, I have been anticipated by Prof. Crofton in his valuable communication to the Society at the April Meeting. It presented itself to me independently on resuming the subject of the present paper in December last. For the sake of subsequent reference, it may be convenient to reproduce it here. It is

$$(\phi D)^n uv = v(\phi D)^n u + n\phi Dv \,.\, (\phi D)^{n-1} u$$

$$+ \frac{n(n-1)}{1\,.\,2} (\phi D)^2 v \,.\, (\phi D)^{n-2} u + \ldots \qquad \ldots\ldots\ldots\ldots (k).$$

It may be compared with Bronwin's Theorem (g), *Phil. Trans*, 1851,

$$v(\phi D)^n u = (\phi D)^n uv - n(\phi D)^{n-1} u\phi Dv + \frac{n\,.\,n-1}{1\,.\,2} (\phi D)^{n-2} u (\phi D)^2 v - \ldots,$$

and either may be deduced from the other.

As in Bronwin's theorems, in the memoir referred to, so in the theorems given in this paper, the operators $D\phi$ or ϕD may be replaced by

$$D\phi + \lambda, \quad \phi D + \lambda,$$

where λ is any function of x; so that

$$(D\phi + \lambda)^2 u = D^2 u\phi^2 + 2\lambda Du\phi + \lambda^2 u,$$

$$(\phi D + \lambda)\, u = (\phi D)^2 u + 2\lambda\phi Du + \lambda^2 u,$$

and so on.

As Hargreave's Theorem II. (*Phil. Trans.*, 1848) is related to his Theorem I., in x being replaced by D, and D by $-x$,* so is my next theorem related to the theorem (g) in the former part of this paper.

[* The reference here, and subsequently, to Hargreave's substitutions was not suggested by Prof. Crofton's use of them in the two concluding paragraphs of his memoir (*ante*, p. 133), which were, in fact, additions made by Prof. Crofton subsequent to the reading of the present paper on June 9th. Prof. Crofton's use of the substitutions was, I understand from him, equally independent of my reference to them.]

If $\phi(x)$, $\chi(x)$ are any rational algebraic functions, and n a positive integer,

$$x^n\phi^n(D)\,\chi(D)\,u = \chi(D)\,[x^n\phi^n(D)]\,u - n\chi'(D)\,\phi(D)\,[x^{n-1}\phi^{n-1}(D)]\,u$$
$$+\frac{n(n-1)}{1.2}\,\frac{d}{dD}[\chi'(D)\,\phi^2(D)]\,[x^{n-2}\phi^{n-2}(D)]\,u - \dots$$
$$\dots + (-1)^n\,\frac{d^{n-1}}{dD^{n-1}}\,[\chi'(D)\,\phi^n(D)]\,u \quad \dots\dots\dots\dots(l);$$

or, more generally, if $f(x)$ is a function developable in positive integer powers of x, and if $\overline{D}^r u = x^r\phi^r(D)\,u$,

$$f(\overline{D})\,\chi(D)\,u = \chi(D)\,f(\overline{D})\,u - \chi'(D)\,\phi(D)\,f'(\overline{D})\,u$$
$$+\frac{1}{1.2}\,\frac{d}{dD}\,[\chi'(D)\,\phi^2(D)]\,f''(\overline{D})\,u - \dots \quad \dots(m).$$

To prove the formula (l), which is well known to hold for $n = 1$, in its generality, assume that it holds for all positive integers up to n inclusive; then, substituting

$$\phi(D)\,u \text{ for } u,$$

multiplying by x, and observing that

$$x\chi(D)\,v = \chi(D)\,(xv) - \chi'(D)\,v,$$
$$x\chi'(D)\,\phi(D)\,v = \chi'(D)\,\phi(D)\,(xv) - \frac{d}{dD}[\chi'(D)\,\phi(D)]\,v,$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$x\,\frac{d^{r-1}}{dD^{r-1}}\,[\chi'(D)\,\phi^r(D)]\,v = \frac{d^{r-1}}{dD^{r-1}}\,[\chi'(D)\,\phi^r D]\,(xv) - \frac{d^r}{dD^r}\,[\chi'(D)\,\phi^r(D)]\,v,$$

there results

$$x^{n+1}\phi^{n+1}(D)\,\chi(D)\,u = \chi(D)\,[x^{n+1}\phi^{n+1}(D)\,u] - \chi'(D)\,[x^n\phi^{n+1}(D)]\,u$$
$$-n\,[\chi'(D)\,\phi(D)]\,[x^n\phi^n(D)\,u] + n\,\frac{d}{dD}\,[\chi'(D)\,\phi(D)]\,[x^{n-1}\phi^n(D)\,u]$$
$$+n_1\,\frac{d}{dD}\,[\chi'(D)\,\phi^2(D)]\,[x^{n-1}\phi^{n-1}(D)\,u]$$
$$-n_1\,\frac{d^2}{dD^2}\,[\chi'(D)\,\phi^2(D)]\,[x^{n-2}\phi^{n-1}(D)\,u]$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$+(-1)^r\,n_{r-1}\,\frac{d^{r-1}}{dD^{r-1}}\,[\chi'(D)\,\phi^r(D)]\,[x^{n-r+1}\phi^{n-r+1}(D)\,u]$$
$$-(-1)^r\,n_{r-1}\,\frac{d^r}{dD^r}\,[\chi'(D)\,\phi^r(D)]\,[x^{n-r}\phi^{n-r+1}(D)\,u] + \dots$$

Now, by the assumption that the development (l) holds for all values of r not exceeding n, giving therein $\chi(D)$ the particular form $\phi(D)$,

$$x^r\phi^{r+1}(D)\,u = \phi(D)\,(x^r\ldots) - r\phi'(D)\,\phi(D)\,(x^{r-1}\ldots) + r_1\frac{d}{dD}\left[\phi'(D)\,\phi^2(D)\right](x^{r-2}\ldots) - \ldots$$

($x^r\ldots$) being an abbreviation of $x^r\phi^r(D)\,u$, so that

$$x^{n+1}\phi^{n+1}(D)\,\chi(D)\,u = \chi(D)\,(x^{n+1}\ldots) - n\chi'(D)\,\phi(D)(x^n\ldots) + n_1\frac{d}{dD}\left[\chi'(D)\,\phi^2(D)\right](x^{n-1}\ldots) - \ldots$$
$$\ldots + (-1)^{r-1}n_r\frac{d^r}{dD^r}\left[\chi'(D)\,\phi^{r+1}(D)\right](x^{n-r}\ldots) + \ldots$$

$$\begin{array}{ll}
+\chi'(D) & \left\{-\phi(D)(x^n\ldots) + n\phi'(D)\,\phi(D)(x^{n-1}\ldots) - n_1\frac{d}{dD}\left[\phi'(D)\,\phi^2(D)\right](x^{n-2}\ldots)\ldots + (-1)^{r-1}n_{r-1}\frac{d^{r-1}}{dD^{r-1}}\left[\phi'(D)\,\phi^r(D)\right](x^{n-r}\ldots) + \ldots\right\} \\
+n\frac{d}{dD}\left[\chi'(D)\,\phi(D)\right] & \left\{\phi(D)(x^{n-1}\ldots) - (n-1)\,\phi'(D)\,\phi(D)\,(x^{n-2}\ldots)\ldots + (-1)^{r-1}(n-1)_{r-2}\frac{d^{r-2}}{dD^{r-2}}\left[\phi'(D)\,\phi^{r-1}(D)\right](x^{n-r}\ldots) + \ldots\right\} \\
+n_1\frac{d^2}{dD^2}\left[\chi'(D)\,\phi^2(D)\right] & \left\{-\phi(D)(x^{n-2}\ldots)\ldots + (-1)^{r-1}(n-2)_{r-3}\frac{d^{r-3}}{dD^{r-3}}\left[\phi'(D)\,\phi^{r-2}(D)\right](x^{n-r}) + \ldots\right\} \\
\ldots & \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots \\
+n_{r-1}\frac{d^r}{dD^r}\left[\chi'(D)\,\phi^r(D)\right] & \left\{+(-1)^{r-1}\qquad\qquad \phi(D)\,(x^{n-r}) + \ldots\right\} \\
\ldots & \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots
\end{array}$$

But by Theorem (a) (*Proceedings*, Vol. XI., p. 108), or by theorem (g) above, with v replaced by ϕ, and u by χ', the coefficient of ($x^{n-1}\ldots$) in this development, excluding the first line, is equal to

$$n\frac{d}{dD}\left[\chi'(D)\,\phi^2(D)\right];\ \text{ that of } (x^{n-2}\ldots) \text{ to } -n_1\frac{d^2}{dD^2}\left[\chi'(D)\,\phi^3(D)\right]\ldots;\ \text{that of } (x^{n-r}\ldots) \text{ to } (-1)^{r-1}n_{r-1}\frac{d^r}{dD^r}\left[\chi'(D)\,\phi^{r+1}(D)\right]\ldots;$$

so that, adding in the first line, the whole development is what (l) becomes when n is increased to $n+1$.

Precisely as the theorem (i) was obtained from (g), so from (l) it follows that

$$x^n\phi^n(D)\chi(D)u-nx^{n-1}\phi^{n-1}(D)\phi'(D)\chi(D)u+n(n-1)\phi^{n-2}(D)\phi'^2(D)\chi(D)u-\dots$$
$$=\chi(D)\,[x^n\phi^n(D)]\,u-n\frac{d}{dD}[\chi(D)\,\phi(D)][x^{n-1}\phi^{n-1}(D)]\,u$$
$$+\frac{n(n-1)}{1\,.\,2}\,\frac{d^2}{dD^2}[\chi(D)\,\phi^2(D)]\,[x^{n-2}\phi^{n-2}(D)]\,u-\dots \qquad \dots\dots(j).$$

Much more readily is it proved that

$$[x\phi(D)]^n\chi(D)\,u=\chi(D)\,[x\phi(D)]^n\,u-n\phi(D)\,\frac{d\chi(D)}{dD}[x\phi(D)]^{n-1}u$$
$$+n_1\left[\phi(D)\frac{d}{dD}\right]^2\chi(D)\,[x\phi(D)]^{n-2}u\dots+(-1)^n\left[\left\{\phi(D)\frac{d}{dD}\right\}^n\chi(D)\right]u$$
$$\dots\dots\dots(n).$$

In fact, assuming it as proved for any one value of n, by operating on both sides with $x\phi(D)$, and observing that

$$x\phi(D)\left(\phi(D)\frac{d}{dD}\right)^r\chi(D)=\left(\phi(D)\frac{d}{dD}\right)^r\chi(D)\,x\phi D-\left(\phi(D)\frac{d}{dD}\right)^{r+1}\chi(D),$$
$$\begin{aligned} n+1 &= n+1,\\ n_1+n &= (n+1)_1,\\ n_2+n_1 &= (n+1)_2,\\ \dots & \quad \dots \quad \dots, \end{aligned}$$

it at once appears that the form holds for the next higher value of n.

The development (n) admits of a generalization similar to those above, viz.,

$$f[x\phi(D)]\chi(D)u=\chi(D)f[x\phi(D)]\,u-\phi(D)\,\frac{d}{dD}\chi(D)f'[x\phi(D)]\,u$$
$$+\frac{1}{1\,.\,2}\left[\phi(D)\,\frac{d}{dD}\right]^2\chi(D)f''[x\phi(D)]\,u-\dots+\dots \qquad \dots\dots\dots(o).$$

Among the applications of the Theorems contained in the two parts of this paper, I shall confine myself at present to the extension of the number of integrable forms of Linear Differential Equations to which they naturally lead.

Hargreave showed in his Memoir on this subject (*Phil. Trans.*, 1848), that the general equation of the second order

$$\phi(x)\,D^2u+2\psi(x)\,Du+\chi(x)\,u=X \qquad \dots\dots\dots\dots\dots(1)$$

is integrable, if $\quad \chi(x)=2\psi'(x)-\phi''(x),$

by his empirical method of substitution, viz., D for x and $-x$ for D; thereby reducing the solution to that of Boole's equation

$$x\lambda(D)\,u+\mu(D)\,u=X.$$

Otherwise, by the substitution of

$$D^2\phi u-2D\phi'u+\phi''u \text{ for } \phi D^2u,$$

and $\quad 2D\psi u-2\psi'u$ for $2\psi Du$,

as was pointed out by Bronwin, the equation (i) is transformed into

$$D\{D\phi u+2(\psi-\phi')u\}+\{\chi(x)-2\psi'(x)+\phi''(x)\}u=X,$$

which at once reduces to the equation of the first order,

$$D\phi u+2[\psi(x)-\phi'(x)]u=D^{-1}X,$$

if $$\chi(x)=2\psi'(x)-\phi''(x).$$

This method may be said to be a treatment of the question so as to apply Theorem (g) for the well-known case of $n=1$.

Now, treating the same question so as to apply that theorem for the case $n=2$:

$$\phi(x)D^2u=D^2[\phi(x)u]-2\phi'(x)Du-\phi''(x)u,$$

by which substitution, and writing $\phi(x)w$ for u,

$$\phi(x)D^2u+2\psi(x)Du+\chi(x)u=X$$

is transformed into

$$D^2\phi^2w+2(\psi-\phi')D\phi w+(\chi-\phi'')\phi w=X,$$

ϕ^2, ψ ... χ being, for shortness, written for $[\phi(x)]^2$, $\psi(x)$ ... $\chi(x)$.

Multiply by $$v=\phi^{-1}\exp.\int\psi\phi^{-1}dx,$$

giving $$v'\phi=(\psi-\phi')\phi^{-1}\exp.\int\psi\phi^{-1}dx$$
$$=(\psi-\phi')v,$$

and $$Dv'\phi^2=[(\psi'-\phi'')\phi+(\psi-\phi')\psi]v;$$

then, if $$\chi(x)=\psi'(x)+[\psi(x)-\phi'(x)]\psi(x)\phi^{-1}(x)$$
$$=[\psi(x)+\phi(x)D][\psi(x)\phi^{-1}(x)],$$

the equation to be solved takes the form

$$vD^2\phi^2w+2v'\phi D\phi w+Dv'\phi^2.w=vX,$$

and, by theorem (g),

$$u=\phi w=\exp.\int-\psi(x)\phi^{-1}(x)dx.D^{-2}[\phi^{-1}(x)\exp.\int\psi(x)\phi^{-1}(x)dx.X],$$

viz., this is the solution of

$$\phi(x)D^2u+2\psi(x)Du+[\psi(x)+\phi(x)D][\psi(x)\phi^{-1}(x)]u=X\ldots(2).$$

Another soluble form of the same equation (1) may be found by treating it so as to apply Theorem (k),

$$\phi D^2u=D\phi Du-\phi'Du,$$

by which substitution and multiplication by ϕ the equation is transformed into $$(\phi D)^2u+(2\psi-\phi')\phi Du+\phi\chi u=\phi X.$$

Again, multiplying by $v=\phi^{-\frac{1}{2}}\exp.\int\psi\phi^{-1}dx$,

which gives $$\phi Dv=(\psi-\tfrac{1}{2}\phi')v,$$
$$(\phi D)^2v=[(\psi'-\tfrac{1}{2}\phi'')\phi+(\psi-\tfrac{1}{2}\phi')^2]v,$$

the equation takes the form

$$v(\phi D)^2u+2(\phi D)v.\phi(D)u+(\phi D)^2v.u=\phi^{\frac{1}{2}}\exp.\int\psi\phi^{-1}dx.X,$$

if $$\chi(x) = [\psi(x)+\phi(x)D]\,\psi(x)\,\phi^{-1}(x) - \tfrac{1}{4}[\phi'(x)+2\phi(x)D]\,\phi'(x)\,\phi^{-1}(x)$$
$$= [\psi(x)+\tfrac{1}{2}\phi'(x)+\phi(x)D]\,[\psi(x)-\tfrac{1}{2}\phi'(x)]\,\phi^{-1}(x).$$

This condition being fulfilled,

$$u=\phi^{\frac{1}{2}}(x)\exp.\int\psi(x)\,\phi^{-1}(x)\,dx\,.\,D^{-1}\phi^{-1}(x)\,D^{-1}[\phi^{-\frac{1}{2}}(x)\,X\int\psi(x)\,\phi^{-1}(x)\,dx];$$

viz., this is the solution of

$$\phi(x)D^2u+2\psi(x)Du+[\psi(x)+\tfrac{1}{2}\phi'(x)+\phi(x)D][\psi(x)-\tfrac{1}{2}\phi'(x)]\phi^{-1}(x).u=X.$$

Five soluble cases of the general linear equation of the third order may similarly be found, which time does not allow of my entering into at present. I pass on to consider the application of theorems (l) and (m), in finding soluble cases of the equation

$$x^2\phi(D)\,u+2x\psi(D)\,u+\chi(D)\,u = X \quad\text{..............(3)},$$

or, assuming $$u=\phi(D)\,w,$$

$$x^2\phi^2(D)\,w+2x\psi(D)\,\phi(D)\,w+\chi(D)\,\phi(D)\,w = X,$$

or, what is the same thing,

$$x^2\phi^2(D)\,w+2\psi(D)\,[x\phi(D)\,w]+[\chi(D)-2\psi'(D)]\,\phi(D)\,w = X.$$

Let $$\lambda(D) = \exp.\int -\psi(D)\,\phi^{-1}D\,dD,$$

giving $$\lambda'(D)\,\phi(D) = -\lambda(D)\,\psi(D),$$

and $$\frac{d}{dD}[\lambda'(D)\,\phi^2(D) = \lambda(D)\,[-\phi(D)\,\psi'(D)-\psi(D)\,\phi'(D)+\psi^2(D)]$$
$$= \lambda(D)\,[\chi(D)-2\psi'(D)]\,\phi(D),$$

if $$\chi(D) = [\psi^2(D)+\phi(D)\,\psi'(D)-\psi(D)\,\phi'(D)]\,\phi^{-1}D$$
$$= \left[\psi(D)+\phi(D)\,\frac{d}{dD}\right]\psi(D)\,\phi^{-1}(D).$$

Operating, then, on the equation above with $\lambda(D)$, the result will take the form

$$\lambda(D)\,x^2\phi^2(D)\,w-2\lambda'(D)\phi(D)\,[x\phi(D)\,w]+\frac{d}{dD}[\lambda'(D)\,\phi^2(D)]\,w=\lambda(D)\,X,$$

if $\chi(D)$ has the value just specified; and, by theorem (l),

$$u=\phi(D)w=\phi(D)^{-1}\exp.\int\psi(D)\,\phi^{-1}(D)\,dD\,[x^{-2}\exp.\int-\psi(D)\,\phi^{-1}(D)]\,X;$$

viz., this is the solution of

$$x^2\phi(D)\,u+2x\psi(D)\,u+\left[\psi(D)+\phi(D)\,\frac{d}{dD}\right][\psi(D)\,\phi^{-1}(D)]\,u = X.$$

Similarly, theorem (m) gives another soluble form of (iii.); and then, by giving $\psi(D)$ such forms as render $\lambda(D)$ an interpretable operator, making $\phi(D)$ in succession $D+a$, $D^2+2aD+b^2$..., integrable forms of somewhat greater generality than those obtained by Hargreave in the memoir cited may be obtained.

I may remark that the soluble forms of equations, found as above, differ from those found by Bronwin, through the application of his Theorems in the Memoir referred to (*Phil. Trans.*, 1851), as must be the case, since the theorems themselves are essentially distinct.

Note on a System of Cartesian Ovals passing through Four Points on a Circle. *By* R. A. ROBERTS, B.A.

[*Read June* 9*th*, 1881.]

Let $S \equiv x^2+y^2-k^2 = 0$ be the equation in rectangular coordinates of the circle, and let $a^2-\beta = 0$, where $a \equiv lx+my$, $\beta \equiv px+qy+r$, denote one of the parabolas whose intersection with S determines the four points; then

$$\vartheta^2(a^2-\beta)+2(\vartheta a+\lambda)S+S^2 = 0 \text{.....................} (1),$$

where ϑ and λ are variable parameters, represents a Cartesian oval passing through the four points; for (1) may be written

$$(S+\vartheta a+\lambda)^2-(\lambda^2+\vartheta^2\beta+2\vartheta\lambda a) = 0 \text{ ....} (2),$$

showing that the curve is a Cartesian oval of which

$$\lambda^2+\vartheta^2\beta+2\vartheta\lambda a = 0 \text{ ....} (3),$$

is the double tangent, and the centre of the circle

$$S+\vartheta a+\lambda = 0 \text{ ....} (4)$$

is the triple focus.

From the equation (4) we see that the triple focus lies on the perpendicular to a at the centre of S, and from (3) that the double tangent touches the parabola $a^2-\beta = 0$.

The equation of a circle Σ, having its centre on the axis (the perpendicular from the triple focus on the double tangent), and having double contact with the curve, is evidently, from (2),

$$\left.\begin{aligned} \mu^2+2\mu(S+\vartheta a+\lambda)+\lambda^2+\vartheta^2\beta+2\vartheta\lambda a &= 0 \\ \text{or} \quad (\lambda+\mu)^2+2\vartheta(\lambda+\mu)a+\vartheta^2\beta+2\mu S &= 0 \end{aligned}\right\} \text{ ....} (5),$$

whence it appears that the radical axis of Σ and S touches the parabola $a^2-\beta = 0$.

Hence, when the radius of Σ is given, its centre lies on a fixed circular cubic; for, expressing that $2x'x+2y'y-(x'^2+y'^2+k^2-r^2) = 0$, the radical axis of S and $\Sigma\,\{\equiv (x-x')^2+(y-y')^2-r^2\}$, touches the parabola $a^2-\beta = 0$, we obtain a relation of the form

$$Ax'^2+By'^2+2Hx'y'+(Gx'+Fy')(x'^2+y'^2+k^2-r^2) = 0.$$

When r vanishes, (x', y') is a focus of the curve, and the cubic is the

envelope of circles which cut S orthogonally and have their centres on the parabola $\alpha^2-\beta=0$ (see Casey's "Bicircular Quartics").

This latter result was proposed as a question in the *Educational Times*, July, 1866, by Prof. Sylvester. (See *Educational Times*, August, 1866, for Prof. Cayley's solution of the above; also October, 1866, for a solution by Prof. Crofton.)

Note on certain Symbolic Operators, and their application to the Solution of certain Partial Differential Equations. By J. W. L. Glaisher, M.A., F.R.S.

[*Read June 9th*, 1881.]

1. Poisson's well-known theorem

$$e^{a^2D^2}\phi(x)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-u^2}\phi(x+2au)\,du \quad\ldots\ldots\ldots\ldots\ldots(1),$$

where D denotes $\frac{d}{dx}$, may be proved very simply as follows.

We have
$$\sqrt{\pi}=\int_{-\infty}^{\infty}e^{-u^2}du \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

whence, putting $u-a$ for u, a being a constant,

$$\sqrt{\pi}=\int_{-\infty}^{\infty}e^{-(u-a)^2}du$$
$$=e^{-a^2}\int_{-\infty}^{\infty}e^{-u^2+2au}du,$$

and therefore
$$\sqrt{\pi}\,e^{a^2}=\int_{-\infty}^{\infty}e^{-u^2+2au}du \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Writing aD in place of a, and taking $\phi(x)$ as the subject of operation,

$$\sqrt{\pi}\,e^{a^2D^2}\phi(x)=\int_{-\infty}^{\infty}e^{-u^2+2auD}du\,.\,\phi(x)\quad\ldots\ldots\ldots\ldots\ldots(4)$$
$$=\int_{-\infty}^{\infty}e^{-u^2}\phi(x+2au)\,du.$$

2. This proof is rigorous, for (3), regarded as an equation involving a, is true identically; that is to say, if both members of the equation are expanded in powers of a, we have

$$\sqrt{\pi}\left(1+a^2+\frac{a^4}{2!}+\&c.\right)=\int_{-\infty}^{\infty}e^{-u^2}\left(1+2au+\frac{(2au)^2}{2!}+\&c.\right)du\quad\ldots(5),$$

and it is easily seen that the coefficients of the same powers of a on each side of the equation are equal to one another, the terms involving

uneven powers of a on the right-hand side being zero. When therefore a is replaced by aD, the operations indicated by each side of the equation are identical; and thus (4), and therefore also (1), must be true.

3. It may be remarked that (1) may be readily verified by expanding $\phi(x+2au)$ by Taylor's theorem in ascending powers of a, and showing that the coefficient of a^{2n} on the right-hand side of the equation is $\dfrac{\phi^{(2n)}(x)}{n!}$. This method of proving (1), which is that which most naturally presents itself, is, it will be observed, equivalent to actually verifying the equality, term by term, of the series in (5). In the proof given in § 1, the equality of these two series is deduced at once from the formula (2).

4. Professor Crofton's formula,*

$$e^{aD^2}e^{kx^2} = (1-4ak)^{-\frac{1}{2}}\, e^{\frac{kx^2}{1-4ak}},$$

may, as he has remarked himself, be readily deduced from (1); for, putting $\phi(x) = e^{kx^2}$, we find

$$\begin{aligned} e^{aD^2}e^{kx^2} &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \epsilon^{-u^2} e^{k(x+2\sqrt{a}u)^2}\,du \\ &= \frac{1}{\sqrt{\pi}}\, e^{kx^2}\int_{-\infty}^{\infty} e^{-u^2+4akw^2+4\sqrt{a}\,kxu}\,du \\ &= \frac{1}{\sqrt{\pi}}\, e^{kx^2}\,\frac{\sqrt{\pi}}{\sqrt{(1-4ak)}}\, e^{\frac{4ak^2x^2}{1-4ak}} = (1-4ak)^{-\frac{1}{2}}\, e^{\frac{kx^2}{1-4ak}}. \end{aligned}$$

5. The formula may, however, be proved somewhat differently as follows. Writing kx^2 for a^2 in (3), we have

$$\sqrt{\pi}\, e^{kx^2} = \int_{-\infty}^{\infty} e^{-u^2+2\sqrt{k}\,xu}\,du.$$

Now
$$\phi(D)\, e^{mx} = \phi(m)\, e^{mx},$$

and therefore
$$\begin{aligned} \sqrt{\pi}\, e^{aD^2} e^{kx^2} &= \int_{-\infty}^{\infty} e^{-u^2} e^{aD^2} e^{2\sqrt{k}\,xu}\,du \\ &= \int_{-\infty}^{\infty} e^{-u^2} e^{a\cdot 4ku^2} e^{2\sqrt{k}\,xu}\,du \\ &= \int_{-\infty}^{\infty} e^{-(1-4ak)u^2+2\sqrt{k}\,xu}\,du \\ &= \frac{\sqrt{\pi}}{\sqrt{(1-4ak)}}\, e^{\frac{kx^2}{1-4ak}}. \end{aligned}$$

---

* *Philosophical Transactions*, 1870, pp. 186, 187.

6. The method of proof employed in the last section shows that in general, $$\phi(D)\, e^{kx^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2+2x\sqrt{k}\,u}\, \phi(2\sqrt{k}\,u)\, du;$$

or, transforming the integral, that

$$\phi(D)\, e^{kx^2} = \frac{1}{2\sqrt{(k\pi)}} \int_{-\infty}^{\infty} e^{-\frac{v^2}{4k}+xv}\, \phi(v)\, dv.$$

7. Poisson's theorem enables us to express $e^{a^2D^2}\phi(x)$ as a single definite integral, and it seems worthy of notice that by the repeated application of this theorem we may express $e^{a^4D^4}\phi(x)$ as a double definite integral, $e^{a^8D^8}\phi(x)$ as a triple definite integral, and so on.

For $$e^{a} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2+2\sqrt{a}\,v}\, dv,$$

whence $$e^{a^4D^4}\phi(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2+2a^2D^2v}\, dv \,.\, \phi(x)$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2}\, dv \,.\, \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2+2\sqrt{2}aD\sqrt{v}\,u}\, du \,.\, \phi(x)$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-u^2-v^2}\, \phi(x+2^{1+\frac{1}{2}}auv^{\frac{1}{2}})\, du\, dv;$$

and, similarly, we find

$$e^{a^8D^8}\phi(x) = \frac{1}{\pi^{\frac{3}{2}}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-u^2-v^2-w^2}\, \phi(x+2^{1+\frac{1}{2}+\frac{1}{4}}auv^{\frac{1}{2}}w^{\frac{1}{4}})\, du\, dv\, dw \quad \text{...(6)},$$

$e^{a^{16}D^{16}}\phi(x)$

$$= \frac{1}{\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-u^2-v^2-w^2-t^2}\, \phi(x+2^{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}auv^{\frac{1}{2}}w^{\frac{1}{4}}t^{\frac{1}{8}})\, du\, dv\, dw\, dt,$$

and so on, the general law being evident.

These results may be verified by expanding the function subject to the sign ϕ, by Taylor's theorem, in ascending powers of a, and integrating term by term. Taking for example (6), it is easy to see that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-u^2-v^2-w^2}\, (uv^{\frac{1}{2}}w^{\frac{1}{4}})^n\, du\, dv\, dw,$$

viz., $$\int_{-\infty}^{\infty} e^{-u^2}u^n du \times \int_{-\infty}^{\infty} e^{-v^2}v^{\frac{1}{2}n}\, dv \times \int_{-\infty}^{\infty} e^{-w^2}w^{\frac{1}{4}n}\, dw,$$

is equal to zero unless n is a multiple of 8, and to verify that, when n is a multiple of 8, the coefficient of a^n on the right-hand side of the equation (6) is equal to $\dfrac{\phi^{(n)}(x)}{(\frac{1}{8}n)!}$.

8. If, now, in the known formula

$$e^{-a} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2 - \frac{a^2}{4u^2}} \, du \quad \dots\dots\dots\dots (7),$$

we substitute $a\sqrt{D}$ for a, and operate upon $\phi(x)$, we find

$$e^{-a\sqrt{D}} \phi(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2 - \frac{a^2}{4u^2} D} \, du \, . \, \phi(x) \quad \dots\dots\dots\dots (8),$$

and therefore $$e^{-a\sqrt{D}} \phi(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} \phi\left(x - \frac{a^2}{4u^2}\right) du \quad \dots\dots\dots\dots (9).$$

In connexion with this result, it is especially to be noticed that the formula (7), unlike (3) in § 1, cannot be proved by expanding both sides of the equation in powers of a, and verifying the equality of the coefficients, for the integral involves a only through a^2, and if it is expanded in powers of a^2 the coefficients are infinite.

The substitution of $a\sqrt{D}$ for a has therefore not been shown to be justifiable, for $e^{a\sqrt{D}}$ can be interpreted only as indicating the performance of the operations

$$1 - aD^{\frac12} + \frac{a^2 D}{2\,!} - \frac{a^3 D^{\frac32}}{3\,!} + \&c.,$$

and it does not appear that the operations indicated by the integral in (8) are identical with these.

It will be noticed that, in (9), if the sign of a is changed, the left-hand member becomes $e^{a\sqrt{D}} \phi(x)$, but the right-hand side remains unaffected.

9. By the repeated application of (8), it can be shown that

$$e^{-aD^{\frac14}} \phi(x) = \left(\frac{2}{\sqrt{\pi}}\right)^2 \int_0^\infty \int_0^\infty e^{-u^2 - v^2} \phi\left(x - \frac{a^4}{2^4 u^4 v^2}\right) du \, dv,$$

$$e^{-aD^{\frac18}} \phi(x) = \left(\frac{2}{\sqrt{\pi}}\right)^3 \int_0^\infty \int_0^\infty \int_0^\infty e^{-u^2 - v^2 - w^2} \phi\left(x - \frac{a^8}{2^8 u^8 v^4 w^2}\right) du \, dv \, dw,$$

and so on, the general law being evident.

If, in the first of these formulæ, we replace a by $-a$, ai or $-ai$, the exponent on the left-hand side becomes $aD^{\frac14}$, $aiD^{\frac14}$, or $-aiD^{\frac14}$, but the right-hand side remains unaltered; similarly, in the second formula, there are eight values of the exponent on the left-hand side which correspond to the same value of the right-hand member, and so on.

The remainder of this note relates to the application of the symbolic formulæ, which have been considered in this and the preceding sections, to the expression of the solutions of certain partial differential equations.

10. The linear partial differential equation

$$\frac{dz}{dx} = a\,\frac{dz}{dy}$$

may, as is well known, be solved as follows.

Let α denote $a\,\dfrac{d}{dy}$, then the differential equation is

$$\frac{dz}{dx} = \alpha z,$$

and therefore $$z = Ce^{\alpha x} = e^{ax\frac{d}{dy}}\,\phi(y)$$
$$= \phi(y+ax).$$

The solution in a form involving an arbitrary function is thus obtained by means of the symbolic expression for the series in Taylor's theorem.

In the same manner we find that the solution of the differential equation $$\frac{dz}{dx} = a^2\,\frac{d^2z}{dy^2} \qquad\text{.............................(10)}$$

is $$z = e^{a^2x\frac{d^2}{dy^2}}\,\phi(y) \qquad\text{......... . ...(11),}$$

and therefore, by (1),
$$z = \int_{-\infty}^{\infty} e^{-u^2}\,\phi(y+2a\sqrt{x}\,u)\,du \qquad\text{..................(12);}$$

so that, in this case, Poisson's theorem enables us to express the solution in a form involving an arbitrary function.

Similarly, the solution of the differential equation

$$\frac{dz}{dx} = a^4\,\frac{d^4z}{dy^4}$$

may, by § 7, be expressed in the form

$$z = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-u^2-v^2}\,\phi\left(y+2^{\frac{3}{2}}ax^{\frac{1}{4}}uv^{\frac{1}{2}}\right)du\,dv\,;$$

and we obtain also solutions, in which z is equated to an expression involving an arbitrary function, of the differential equations

$$\frac{dz}{dx} = a^8\,\frac{d^8z}{dy^8},\ \ \&c.$$

11. Returning to the differential equation (10), if we denote $\dfrac{1}{a^2}\,\dfrac{d}{dx}$ by β, the equation becomes

$$\frac{d^2z}{dy^2} = \beta z,$$

and therefore $$z = Ae^{y\sqrt{\beta}} + Be^{-y\sqrt{\beta}}$$
$$= e^{\frac{y}{a}\sqrt{\frac{d}{dx}}}f(x) + e^{-\frac{y}{a}\sqrt{\frac{d}{dx}}}F(x) \qquad\text{..................(13),}$$

and either of these two terms leads, by § 8, to the formula

$$z = \int_0^\infty e^{-u^2} f\left(x - \frac{y^2}{4a^2u^2}\right) du \dots\dots\dots\dots\dots\dots (14).$$

From this expression for z, we find

$$\frac{d^2z}{dy^2} = \int_0^\infty e^{-u^2} f''\left(x - \frac{y^2}{4a^2u^2}\right) \frac{y^2}{4a^4u^4}\, du - \int_0^\infty e^{-u^2} f'\left(x - \frac{y^2}{4a^2u^2}\right) \frac{1}{2a^2u^2}\, du,$$

$$\frac{dz}{dx} = \int_0^\infty e^{-u^2} f'\left(x - \frac{y^2}{4a^2u^2}\right) du,$$

and, substituting these expressions in the differential equation,

$$\int_0^\infty e^{-u^2} f''\left(x - \frac{y^2}{4a^2u^2}\right) \frac{y^2}{4a^4u^4}\, du$$

$$= \int_0^\infty e^{-u^2} f'\left(x - \frac{y^2}{4a^2u^2}\right) \frac{1}{2a^2u^2}\, du + \frac{1}{a^2}\int_0^\infty e^{-u^2} f'\left(x - \frac{y^2}{4a^2u^2}\right) du.$$

Integrating by parts, the left-hand member

$$= \frac{1}{a^2}\left[\frac{e^{-u^2}}{2u} f'\left(x - \frac{y^2}{4a^2u^2}\right)\right]_0^\infty + \frac{1}{a^2}\int_0^\infty f'\left(x - \frac{y^2}{4a^2u^2}\right)\left(e^{-u^2} + \frac{1}{2u^2}e^{-u^2}\right) du,$$

so that the differential equation is satisfied if $\frac{e^{-u^2}}{2u} f'\left(x - \frac{y^2}{4a^2u^2}\right)$ vanishes when $u = 0$. We observe also that f must be such that both the integrals in the expression for $\frac{d^2z}{dy^2}$ are finite.

Subject to these conditions, the formula (14) satisfies the differential equation; and it may be remarked, in reference to these conditions, that it seems pretty clear that they will generally be satisfied whenever the form of the function f is such that the integral in (14) is finite.

12. The formula (12) satisfies the differential equation unconditionally, for we have

$$\frac{d^2z}{dy^2} = \int_{-\infty}^\infty e^{-u^2} \phi''(y + 2a\sqrt{x}\,u)\, du,$$

$$\frac{dz}{dx} = \int_{-\infty}^\infty e^{-u^2} \phi'(y + 2a\sqrt{x}\,u) \frac{au}{\sqrt{x}}\, du,$$

and, integrating the expression for $\frac{d^2z}{dy^2}$ by parts, we find that it

$$= \left[\frac{e^{-u^2}\phi'(y + 2a\sqrt{x}\,u)}{2a\sqrt{x}}\right]_{-\infty}^\infty + \int_{-\infty}^\infty e^{-u^2}\phi'(y + 2a\sqrt{x}\,u) \frac{u}{a\sqrt{x}}\, du,$$

of which the first term is zero owing to the presence of the factor e^{-u^2}.

13. In his "Examples of the Processes in the Differential and Integral Calculus" (Cambridge, 1841, p. 350), Gregory gives the solution of the differential equation (10) in the two symbolic forms (11) and (13); from the former he deduces that

$$z = \phi(y) + a^2 x \phi''(y) + \frac{a^4 x^2}{2!} \phi^{\mathrm{iv}}(y) + \&c.,$$

and from the latter, by putting

$$f(x) + F(x) = \psi(x), \quad \left(\frac{d}{dx}\right)^{\frac{1}{2}} \{f(x) - F(x)\} = \chi(x),$$

he deduces that

$$z = \psi(x) + \frac{1}{2!}\frac{y^2}{a^2}\psi'(x) + \frac{1}{4!}\frac{y^4}{a^4}\psi''(x) + \&c.$$

$$+ \frac{y}{a}\chi(x) + \frac{1}{3!}\frac{y^3}{a^3}\chi'(x) + \frac{1}{5!}\frac{y^5}{a^5}\chi''(x) + \&c.$$

In reference to these two solutions in series, he remarks that "it seems anomalous that the same equation should admit of two solutions differing so essentially in character that the one contains two arbitrary functions, and the other only one," and he mentions certain considerations "which may serve to explain the difficulty."

It is partly on account of the attention thus directed to the two forms (11) and (13) of the solution of the differential equation, that I have thought it worth while to notice the formula (14), which is derived from (13), and involves only one arbitrary function, as it is curious that both terms of (13) lead by means of (9) to the same expression. It also seemed of interest, having regard to the completeness and rigour with which Poisson's theorem admits of being established, and to the unsatisfactory character of the process by which (9) was obtained, to point out that the former leads to an exact solution of the differential equation, and the latter to a solution which is true only under certain conditions.

14. If the differential equation

$$\frac{dz}{dx} = a^4 \frac{d^4 z}{dy},$$

is integrated by treating $\frac{d}{dx}$ as a constant, the four terms in the solution all lead to the formula

$$z = \int_0^\infty \int_0^\infty e^{-u^2-v^2} \phi\left(x - \frac{y^4}{16a^4u^4v^2}\right) du\, dv,$$

and so on; but the solutions of these differential equations of higher orders present no additional point of interest.

The following presents were made to the Society's Library during the Recess :—

"Mittheilungen der Naturforschenden Gesellschaft in Bern in dem Jahre 1880," Nos. 979—1003; Bern, 1881.

"On Binomial Congruences, comprising an Extension of Fermat's and Wilson's Theorems, and a Theorem of which both are Special Cases," by O. H. Mitchell (from "American Journal of Mathematics," Vol. iii., No. 4): from the Author.

"Observations of Double Stars made at the United States Naval Observatory," by Asaph Hall; Washington, 1881.

"Account of the Operations of the Great Trigonometrical Survey of India," Vol. vi.—The principal Triangulation of the South-east Quadrilateral: from Her Majesty's Secretary of State for India, in Council.

"Educational Times," August, September, October, November, 1881; and "Reprint from the Educational Times," Vol. xxxv.: from the Publishers.

"Crelle," 91° Band, 2es, 3es, 4es Heft; Berlin, 1881.

"Proceedings of the Royal Society," Vol. xxxii., Nos. 214, 215.

"Physical Society. Proceedings," Vol. iv., Pt. iii., May to June, 1881; Vol. iv., Pt. iv., July to October, 1881.

"Bulletin des Sciences Mathématiques et Astronomiques," 2me Série, Tome v., Févr., Mars, Avril, Mai, Juin, 1881.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich," Jahrgang 24, 1es, 2es, 3es, 4es Heft, 1879; Jahrgang 25, 1es, 2es, 3es, 4es Heft, 1880.

"Proceedings of Royal Irish Academy—Science," Vol. iii., Series ii., No. 5, December, 1880; No. 6, April, 1881.

"Proceedings of Royal Irish Academy—Polite Literature and Antiquities," Vol. ii., Series 2, No. 2, December, 1880.

"Transactions of the Royal Irish Academy—Science," Vol. xxviii.; i. (December, 1880), ii. (February, 1881), iii., iv. (February, 1881), v. (March, 1881).

"Transactions of the Royal Irish Academy—Polite Literature and Antiquities," Vol. xxvii., iv. (June, 1881).

"Monatsbericht," April, Mai, Juni, 1881.

"Beiblätter zu den Annalen der Physik und Chemie," Band v., Stück 7, No. 7; St. 8, No. 8; St. 9, No. 9; St. 10, No. 10.

"Jahrbuch über die Fortschritte der Mathematik," elfter Band, Jahrgang 1879, Heft 1, 2; Berlin, 1881.

"Reale Istituto Lombardo—Rendiconti," Serie ii., Vol. xii., 1879.

"Jornal de Sciencias Mathematicas e Astronomicas," publicado pelo Dr. Fr. Gomes Teixeira, Vol. i., 1878; Vol. ii., 1880; Vol. iii., Nos. 1, 2, 3, 1881; Coimbra.

"Verhandlungen der Schweizerischen Naturforschenden Gesellschaft in Brieg," den 13, 14, 15 Sept., 1880 (63 Jahresversammlung).

"Jahresbericht," 1879, 1880; Lausanne, 1881.

"A Determination of the Solar Parallax from Observations of Mars made at the Island of Ascension in 1877," by David Gill, LL.D., 1881 ("Memoirs of Royal Astronomical Society," Vol. xlvi.): from the Author.

"Catalogue of the Literary, Historical, and Scientific Library of the celebrated Mathematician Mr. Michel Chasles, of the Institute of France," to be sold by auction at Paris on the 27th June, 1881, and eighteen following days (Sundays excepted), (with corresponding French title), 3 Fasciculi in 390 8vo. pp.

"Mittheilungen der Mathematischen Gesellschaft in Hamburg," No. 1, Mai, 1881.

"The Mathematical Visitor," edited and published by Artemas Martin, M.A., Vol. i., 1877—81, Erie, Pa., 1881: from the Editor.

"American Journal of Mathematics," Vol. iii., No. 4.

"Measures, Weights, and Moneys of all Nations, and an Analysis of the Christian, Hebrew, and Mahometan Calendars," by W. S. B. Woolhouse, F.R.A.S., 6th edition, 1881.

"On the Adjustment of Mortality Tables, a sequel to former papers on the same subject," by W. S. B. Woolhouse, F.R.A.S.; from the Author.

"Annali di Matematica," Tomo x°, Fasc. 3° (Ott. 1881).

"In Memoriam Dominici Chelini, Collectanea Mathematica," nunc primum edita cura et studio L. Cremona et E. Beltrami; Hoepli, Mediolani, 1881: from the Editors.

The following is a list of the volumes from the late Mr. Henry Warburton's Library, which were presented to the Society by Mr. Howard Elphinstone (see p. 36):—

"Du Calcul des Dérivations," par L. F. A. Arbogast; Strasbourg, 1800.

"Essai sur la Théorie des Nombres," par A. M. Legendre, 2e édition; Paris, 1808.

"Lilawati, or a Treatise on Arithmetic and Geometry," by Bhascara Acharya, translated by J. Taylor, M.D.; Bombay, 1816 (presented to John Pond, Astronomer Royal, by the Translator).

"Histoire de l'Astronomie Ancienne," par M. Delambre, Vols. i., ii.; Paris, 1817.

"Astronomie Théorique et Pratique," par M. Delambre, Vols. i., ii., iii.; Paris, 1814.

"Histoire de l'Astronomie du Moyen Age," par M. Delambre; Paris, 1819.

"Mécanique Analytique," par J. L. Lagrange, nouvelle édition, Tome i., 1811; Tome ii., 1815; Paris.

"Meditationes Analyticæ," ab Edvardo Waring, M.D.; Cantab. 1776.

"The History and Present State of Discoveries relating to Vision, Light, and Colours," by Joseph Priestley, F.R.S.; London, 1772.

J. A. Borelli, "Atrium Physico-mathematicum apertum .... (De Vi Percussionis .... et de Motu Animalium ....);" Lugd. Bat., 1686 (Editio prima Belgica accurante J. Broen).

"The Anti-logarithmic Canon, being a Table of Numbers .... to which is prefixed an Introduction by James Dodson;" London, 1742.

"Miscellanea Philosophico-Mathematica Societatis Privatæ Taurinensis," Tom. i., 1759; Tom. ii., 1760—61; Tom. iii., 1762—65; Tom. iv., 1766—69; Tom. v., 1770—73; Turin.

"Christiani Hugenii Opera reliqua," Vols. i., ii., 1728; Amstelodami.

"Christiani Hugenii Opera varia," Vols. i., ii., 1724; Lugduni Batavorum.

"Leonhardi Euleri Opuscula Analytica," Tom. i., 1783; Tom. ii., 1785; and "Tentamen Novæ Theoriæ Musicæ," 1739 (in one volume); Petropoli.

"Leonhardi Euleri Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes" ....; Lausannæ et Genevæ, 1744.

"Leonhardi Euleri Mechanica, sive Motus Scientia analytice exposita," Tom. i., ii.; Petropoli, 1736.

"Leonhardi Euleri Opuscula varii argumenti" (6 papers); Berolini, 1746.

"G. G. Leibnitii et Joh. Bernoullii Commercium Philosophicum et Mathematicum," Tom. i., ii. (in one Vol.); Lausannæ et Genevæ, 1745.

"Conjectura physica circa propagationem Soni et Luminis, una cum aliis dissertationibus analyticis de numeris amicabilibus, de natura æquationum ac de rectifica-

tione ellipsis" (1750); opusculum Tomus iii., continens novam Theoriam Magnetis" (1751); (all in one volume).

"Geometria Organica, sive descriptio linearum curvarum universalis," auctore C. MacLaurin; Londini, 1720.

"Leon. Euleri Theoria motus corporum solidorum seu rigidorum;" Rostochii, 1765.

"A Treatise of Mechanics, Theoretical, Practical, and Descriptive," by Olinthus Gregory, Vol. i., ii., and Volume of plates; London, 1806.

"Archimedis Opera; Apollonii Pergæi Conicorum Libri iv.; Theodosii Sphærica, Methodo novâ illustrata et succincte demonstrata per Isaacum Barrow;" Londoni, 1675.

"Elements of Mechanical Philosophy," by J. Robison, LL.D.; Vol. i., Dynamics and Astronomy; Edinburgh, 1804.

"A Treatise on Algebra, with Appendix concerning the General Properties of Geometrical Lines," by Colin Maclaurin (4th edition); London, 1779.

"Jacobi Bernoulli ars Conjectandi, Tractatus de Seriebus Infinitis, et Epistola Gallice scripta de ludo Pilæ reticularis;" Basileæ, 1713. "Joh. Bernoulli de motu Musculorum," and "Nic. Bernoulli de usu Artis Conjectandi in Jure." (These five in one volume.)

"Leçons sur le Calcul des Fonctions," nouvelle édition, J. L. Lagrange; Paris, 1806.

"Leon. Euleri Introductio in Analysin Infinitorum," Tom. i., ii.; Lugduni, 1797. (In one volume.)

"Leon. Euleri Elémens d'Algèbre" (Nouvelle édition, par J. G. Garnier), Tome i., ii., 1807; Paris.

"Opticks, or a Treatise of the Reflections, Refractions, Inflections, and Colours of Light," by Sir Isaac Newton, Second edition; London, 1718.

"Cours de Mathématiques," par Charles Bossut, Tome second (Géométrie et application de l'Algèbre à la Géométrie); Paris, 1800 (An. ix.)

"Bija Ganita, or the Algebra of the Hindus," by Edward Strachey; London, 1813.

"Essay on the Resolution of Algebraic Equations," by Giffin Wilson (Phil. Trans., read June, 1799).

"On Self-repeating Series," by H. Warburton (Camb. Phil. Soc. Trans., Vol. ix., Pt. iv., duplicate).

"On Simultaneous Differential Equations of the First Order in which the Number of the Variables exceeds by more than one the Number of the Equation," by G. Boole, F.R.S. (Phil. Trans., read June 19, 1862).

"The Diffusion of Liquids," by T. Graham, F.R.S. (Phil. Trans., Pt. 1 for 1850).

"On the Partition of Numbers, and on Combinations and Permutations" (Camb. Phil. Soc. Trans., Vol. viii., Pt. iv.), by H. Warburton, F.R.S. (interleaved and bound copy).

In a case—"Animal Mechanics," "Hydrostatics," "Elements of Trigonometry" (W. Hopkins); "Elements of Spherical Trigonometry" (De Morgan); "On Probability," "On the Study of Mathematics," and other numbers of Mathematical Works in the Library of Useful Knowledge series.

"On the Application of Liquids formed by the Condensation of Gases as Mechanical Agents," by Sir H. Davy, P.R.S. (Phil. Trans., 1823).

"Origine, trasporto in Italia, primi progressi in essa dell' Algebra: Storia critica di nuove disquisizioni analitiche e metafisiche," abricchita di D. Pietro Cossali, C.R., Vol. i., 1797; Vol. ii., 1799; Parma.

"Isaaci Newtoni Opera quæ exstant omnia," commentariis illustrabat Sam. Horsley, LL.D., R.S.S., Londini; Vols. i., ii., 1779; Vols. iii., iv., 1782; Vol. v., 1785.

"Scriptores Logarithmici . . . ." (F. Maseres), Vols. i., ii., 1791; Vol. iii., 1796; Vol. iv., 1801; Vol. v., 1804; Vol. vi., 1807.

"Bibliographie Astronomique, avec l'Histoire de l'Astronomie depuis 1781 jusqu'à 1802," par Jérôme de la Lande; à Paris, 1803.

"Traité de l'Astronomie Indienne et Orientale," par M. Bailly; Paris, 1787.

"Introduction à l'Analyse des Lignes Courbes Algébriques," par G. Cramer; Genève, 1750.

"The Differential and Integral Calculus. . . .," by Augustus De Morgan (Library of Useful Knowledge); London, 1842.

"Traité élémentaire du Calcul des Probabilités," par S. F. Lacroix; Paris, 1822.

"Jacobi Bernoulli Basileensis Opera," Tom. i., ii.; Genevæ, 1744.

"Johannis Bernoulli Opera omnia," Tome i., ii., iii., iv., Lausannæ et Genevæ, 1742.

"A Treatise on the Rectilinear Motion and Rotation of Bodies, with a description of original experiments relative to the subject," by G. Attwood, F.R.S.; Cambridge, 1784.

"Christiani Hugenii Κοσμοθεωρος;" Hagæ-Comitum, 1699.

"Treatise of Algebra in two books," by W. Emerson; London, 1764.

"Miscellanies, or a Miscellaneous Treatise, containing several Mathematical subjects," by W. Emerson; London, 1776.

"The Theory of the Moon, and on the Perturbations of the Planets," and other Tracts, by J. W. Lubbock, F.R.S.; London, 1833 (in duplicate).

"Miscellanea Mathematica," by C. Hutton, F.R.S.; London, 1775.

"The Diarian Miscellany," by Chas. Hutton (3 Vols.); London, 1775.

"Traité de Mécanique élémentaire," par L. B. Francœur; Paris (An. ix.)

"Cours complet de Mathématiques Pures," par L. B. Francœur; Paris, 1819. Seconde édition, Tomes i., ii.

"Uranographie, ou Traité élémentaire d'Astronomie," par L. B. Francœur, Seconde édition; Paris, 1818.

"Recueil de diverses Propositions de Géométrie," par L. Puissant, Seconde édition; Paris, 1809.

"Réflexions sur la Métaphysique du Calcul Infinitésimal," par M. Carnot; Paris, 1813.

"Essais de Géométrie sur les Plans et les Surfaces Courbes," par S. F. Lacroix, 3me édition; Paris, 1808. And "Traité élémentaire de Trigonométrie rectiligne et sphérique, 1810." And "Elémens de Géométrie à l'usage de l'Ecole Centrale des Quatre-nations," Neuvième édition, 1811. (These three in one volume.)

"A Geometrical Treatise of Conic Sections" (in 4 books), by Rev. Abram Robertson, F.R.S.; Oxford, 1802.

"The Phenomena and Order of the Solar System," by J. P. Nichol, LL.D.; Edinburgh, 1838.

"Complément des Elémens d'Algèbre," par S. F. Lacroix, 4e édition; Paris, 1817.

"Catoptricæ et Dioptricæ Sphæricæ Elementa," auctore D. Gregorio, M.D.; Oxonii, 1695.

"Histoire générale des Mathématiques," par Charles Bossut, Tome i., ii., 1810; Paris.

"Elements of the Conic Sections," by Dr. Rob. Simson; Edinburgh, 1775.

"The First Book of Euclid's Elements, with alterations and familiar notes," by a Member of the University of Cambridge (? Col. Peyronnet Thompson), in usum serenissimæ filiolæ; London, 1830.

"Elementary Principles of the Theories of Electricity and Heat, and Molecular Actions," by Rev. R. Murphy, Pt. i., on Electricity; Cambridge, 1833.

"A Syllabus of the Elementary System of Astronomy" (interleaved).

"A Treatise on Isoperimetrical Problems and the Calculus of Variations," by R. Woodhouse, F.R.S.; Cambridge, 1810.

"A Treatise on Plane and Spherical Trigonometry," by R. Woodhouse, F.R.S.; Cambridge, 1819

"Geometrical Problems . . ., and an Appendix containing the Elements of Plane Trigonometry," by Miles Bland, D.D., F.R.S.; London, 1842.

"A Familiar Introduction to Crystallography, with an Appendix," by Henry James Brooke, F.R.S.; London, 1823.

"The Heat of Vapours, and on Astronomical Refractions," by J. W. Lubbock, F.R.S.; London, 1840.

"When is Venus brightest?" by A. S. Herschel ("Quarterly Journal of Pure and Applied Mathematics," February, 1861).

"On an Approximate and Graphical Rectification of the Circle," by A. S. Herschel (from same, October, 1860).

"Factorial Notation," by H. W. Elphinstone (from same, May, 1857).

"Memoir on a New and Certain Method of ascertaining the Figure of the Earth by means of Occultations of the Fixed Stars," by A. Cagnoli, with Notes and an Appendix by Francis Baily; London, 1819.

"Trigonometry and Double Algebra," by Aug. de Morgan; London, 1849.

APPENDIX.

The remarks on Reversion (p. 3) were merely an abstract of what now appears in Mr. Taylor's *Geometry of Conics*, pp. lxxxvi. and 321—328; compare also the *Quarterly Journal of Mathematics*, Vol. xiv., pp. 25—31.

Prof. Teixeira's Note (p. 14) was a trifling one, though the theorem it contained is interesting, "a kind of Leibnitz's Theorem for Determinants."

Prof. Cayley put it in the following compact form:—

$$\delta_x^n D = \Sigma \frac{\Pi n}{\Pi\alpha \,.\, \Pi\beta \,.\, \Pi\gamma \,.\, \Pi\delta} \cdot \begin{array}{c} \delta_x^\alpha \\ \delta_x^\beta \\ \delta_x^\gamma \\ \delta_x^\delta \end{array} \begin{vmatrix} \phi_1 & \psi_1 & \theta_1 & \omega_1 \\ \phi_2 & \psi_2 & \theta_2 & \omega_2 \\ \phi_3 & \psi_3 & \theta_3 & \omega_3 \\ \phi_4 & \psi_4 & \theta_4 & \omega_4 \end{vmatrix},$$

$$\alpha+\beta+\gamma+\delta=n;$$

where the δ^n_x, &c. apply to the several lines of the determinant. But this is nothing else than $\delta^n_x D = (\delta_1+\delta_2+\delta_3+\delta_4)^n D$,

the δ_1, δ_2, δ_3, δ_4 applying to the several lines of the determinant.

Mr. H. M. Jeffery's paper, referred to, p. 26, is given in *Quarterly Journal of Mathematics*, No. 69, pp. 1—40; there is also a paper by him in No. 70, under the title "On the Stapete-points of Class-Quartics with Quadruple Foci," pp. 158—166.

In the *Revue Maritime et Coloniale*, for November, 1880, is published a paper by Lieut. A. Corrard, in which the author prefaced a discussion on the collision of ships by some considerations of elementary geometry on the uniform rectilinear motion of two vessels. Mr. Merrifield, in his short communication (p. 36), showed that, though the author preferred to use demonstrations dependent upon the straight line and circle only, the question really turned upon the consideration that the line joining their centres envelopes a parabola.

In § 19 (p. 134) of Mr. Crofton's paper, read

$$r^{x(\log x)D}\,F(x) \equiv F(x^r)\,r^{x(\log x)D},$$

and
$$F(D^r) \equiv r^{-D(\log D)x}\,F(D)\,r^{D(\log D)x}.$$

Mr. Emory McClintock, of Milwaukee, the author of a paper entitled "An Essay on the Calculus of Enlargement" (*American Journal of Mathematics*, Vol. ii.), calls attention to the fact that a theorem of some importance, equation (69) of his paper,

$$S^h = 1+hR+\frac{h^2R^2}{1\,.\,2}+\ldots,$$

where
$$R = \log S,$$

[later on, he shows that $Rv = \dfrac{Dv}{D\psi x}$, equation (90),] has been substantially rediscovered by Prof. Crofton (pp. 127—129). "The plan of my work required the postponement of the discussion of differentiation .... I am glad that he (Prof. C.) has so well illustrated the practical utility of the theorem, from a point of view other than that in which the plan of my essay led me to exhibit it." Mr. Crofton admits that the theorem is immediately deducible, and writes, "of course I would have stated this in my paper, had I been aware of it." Speaking very favourably of what appears to him to be "a very original and remarkable essay," Prof. Crofton concludes,—"I do not think the theorem, in the form I have given, can be said to have been given by Mr. McClintock in his essay, while I quite admit that, by putting together his series (69) and one of the values (90), given later on for R, it may be at once deduced."

Take any point P on a curve, O the source of light, OP, PA the incident and reflected rays respectively, PG, PT the normal and tangent to the curve at P; the path of the ray $= OP+PA \equiv v$. In order to discriminate between the maximum and minimum paths, we must ascertain the sign of $\frac{d^2v}{dx^2}$,

$$v = \sqrt{x^2+y^2} + \sqrt{(a-x)^2+y^2}, \quad \frac{OG}{OP} = \frac{GA}{PA},$$

$$\frac{d^2v}{dx^2} = \left(\frac{1}{OP} + \frac{1}{PA}\right) \left\{ \frac{(y-px)^2}{x^2+y^2} + qy \right\}.$$

The object of Mr. H. M. Taylor's question (p. 109) was to get a geometrical value for the second factor.

The following is an abstract of Mr. Merrifield's Note on Ptolemy's Theorem (Euc. vi., D), p. 142:—

If A, B, C, D be four points on a circle, Ptolemy's theorem tells us that the lengths of the lines joining them are subject to the relation

$$AB\,.\,CD + AC\,.\,DB + AD\,.\,BC = 0.$$

But, since the four points lie in a plane, the lines joining them must also be subject to the further relation $V = 0$, where V is the volume of the tetrahedron $ABCD$. The question may be asked, what becomes of the lines which satisfy Ptolemy's relation, when the restriction as to their lying in a plane is removed?

This question is very easily answered with the help of Carnot's expression, which gives the radius of the circumscribed sphere; for, calling this R, we have

$$\begin{aligned} 576\,R^2V^2 \equiv\ & (+AB\,.\,CD + AC\,.\,DB + AD\,.\,BC) \times \\ & (- \qquad\qquad + \qquad\qquad + \qquad\qquad) \times \\ & (+ \qquad\qquad - \qquad\qquad + \qquad\qquad) \times \\ & (+ \qquad\qquad + \qquad\qquad - \qquad\qquad). \end{aligned}$$

If the second side of this equation vanishes, as it does when Ptolemy's equation holds, either R or V must vanish. If R vanishes, the four points of the tetrahedron coalesce in a single point. If V vanishes, the four points lie in a plane, and R takes the value $\frac{0}{0}$, as it ought to do, being only restricted by the condition that it should not be less than the radius of the circle through the four points in plano.

The existence of Ptolemy's relation therefore implies, that the four points must either lie in a plane (and, if so, in a circle), or else be coincident.

In connexion with Mr. Carpmael's paper, we may refer to *Proceedings*,

Vol. vii., p. 235, adding thereto a reference to the *Quarterly Journal of Mathematics*, Vol. xi. (1871), pp. 26—37, where is given a "General Solution and Extension of the Problem of the 15 School Girls," by Andrew Frost.

Mr. McColl has printed in the *Philosophical Magazine* (Jan. 1881, p. 40) a paper entitled "Implicational and Equational Logic," after the method of his communications to the *Proceedings*.

The following two articles on Linkages have been noted :—"Linkages for X^m," by F. T. Freeland (*American Journal of Mathematics*, Vol. iii., No. 4); and "Note sur le système articulé du Colonel Peaucellier," par M. D'Ocagne, in the *Nouvelles Annales* (Oct. 1881, pp. 456, &c.).

In reference to a letter from Signor Brioschi, mentioned in the last volume of the *Proceedings* (xi., p. 157), Sir J. Cockle cites three papers by Signor Brioschi (*Annali de Matematica*, Tom. ix., p. 11, Tom. x., p. 1 and p. 4), and a paper by M. Hermite (*ib.*, Tom. ix., p. 21); and further remarks that references to all, or nearly all, the papers bearing on the subject will be found in Prof. Cayley's recent memoir "On the Schwarzian Derivative," &c. (*Cambridge Philosophical Society Transactions*, Vol. xiii., p. 5).

Herr W. Schlötel, of Strassburg, in a letter to the Secretaries, dated January 21, 1881, "in order to secure priority" as against Messrs. C. S. Peirce and McColl, refers to papers communicated by him to the *Augsburger Allgemeine Zeitung*, in 1868, 1871, and 1876.

Mr. T. Muir has contributed to the *Quarterly Journal of Mathematics*, Vol. xviii., No. 70, a very complete "List of Writings on Determinants" (*cf. Proceedings London Mathematical Society*, Vol. xi., p. 157).

The Subscription List of the De Morgan "Memorial Medal" Fund having been closed, it may be stated here that the money obtained has been invested by the Treasurer as £104. 19*s*. 3*d*. Reduced 3 per cent. Consols.

A brief allusion is due to the losses the Society has sustained in the past session by the deaths of two of its members, each of whom surpassed the threescore years and ten.

In M. Chasles, we lost our first Foreign member. No one could value this title more highly than M. Chasles did.

For notices of his life and works, we refer to the funeral speeches (extracted from the *Comptes Rendus*), published in the *Bulletin des Sciences Mathématiques et Astronomiques*, Tome iv., December 1880, by M. Bertrand (pp. 433—435), and M. Bouquet (pp. 435, 436).

In the same number of the *Bulletin* is an appreciative notice by M. Darboux (pp. 436—442); in *Nature* (Vol. xxiii., No. 584), January 6, 1881, there is also a brief memoir, from which we make a few extracts.

Michel Chasles was born at Épernon (Eure-et-Loire), November 15th, 1793, and entered the École Polytechnique in 1812. He was present at the defence of Paris, in 1814, and reentered the above-named school in 1815. In 1839 Chasles was elected a corresponding member of the Academy, was "decorated" the same year, and in 1841 was made "Professeur de Machines et de Géodésie" at the École Polytechnique. This chair he held for ten years. Meanwhile he was elected to the newly created chair of Modern Geometry in 1846. The following are a few among the many honours Chasles obtained:—in 1851 he was elected a member of the Academy; in 1854 he became a Foreign Member of the Royal Society; in 1865 he was awarded the Copley Medal, and in April 1867 he was elected our first Foreign Member, a distinction which he for some time enjoyed alone. "M. Chasles a poursuivi son œuvre sans interruption depuis sa sortie du Lycée jusqu' à l'âge de quatre-vingt-sept ans. Soixante-huit années séparent la première note de l'élève Chasles, insérée dans la *Correspondance sur l'Ecole Polytechnique*, du dernier mémoire présenté à l'Académie des Sciences. Tous les géomètres, sans distinction de nationalité ni d'école, se sont inclinés devant ce vénérable vieillard; tous ont admiré sa puissance d'invention, sa fécundité, que l'âge semblait rajeunir, son ardeur, et son zèle, continués jusqu' aux derniers jours." M. Chasles died December 18th, 1880. "La vie de M. Chasles a été heureuse et simple; il a trouvé dans la science, avec les plus grandes joies, une gloire qui sera immortelle, et dans la vive affection de ses amis, dans leur assiduité empressée aux réunions où il les conviait avec une grâce si aimable, dans leur respectueuse déférence en toute circonstance, la consolation de sa vieillesse."

We give here the dates and titles of M. Chasles's larger works:—

1. "Aperçu historique sur l'origine et le développement des méthodes en Géométrie .... suivi d'un mémoire .... sur deux principes généraux .... et l'Homographie," 1837, reprinted verbatim in 1875.
2. "Traité de Géométrie supérieure," 1852, reprinted in 1880.
3. "Les Trois Livres de Porismes d'Euclide, rétablis pour la première fois, d'après la Notice et les Lemmes de Pappus, et conformément au sentiment de R. Simson sur la Forme des Enoncés de ces Propositions," 1863.
4. "Traité des Sections Coniques, faisant suite au Traité de Géométrie supérieure," Vol. i., 1865.
5. "Rapport sur les progrès de la Géométrie," 1870.

The number of the smaller papers printed in Chasles's lifetime, we think, exceeded 200; and the number of his scientific MSS., presented to the Academy of Sciences by M. Favre, is stated to be 113 (*Nature*, Jan. 5, 1882, No. 636).

M. Chasles was personally known to a few only of our members, but no one who attended our meetings from November 1865 to June 1878,

with any regularity could have failed to have seen Mr. T. Cotterill, who during that time was rarely absent and was a most interested auditor of all papers bearing on plane curves. Of him the late Prof. Clifford wrote (*Mathematical Papers*, p. 42),—"Mr. Cotterill, is, I believe, the first person that ever *saw* a curve of the third class."

The Rev. J. E. B. Mayor, in a notice of "The Cotterells, Cotterills, and Cottrells, of Cambridge" (*Notes and Queries*, November 12, 1881), states that Mr. T. Cotterill was the son of the Rev. Thomas Cotterill,* formerly Fellow of St. John's College, and was born at Lane End, county Stafford, of which place his father was then the minister. Subsequently his father was Perpetual Curate of St. Paul's, Sheffield, and young Cotterill was educated at the Grammar School of that town. He was entered as a pensioner at St. John's College, Cambridge, January 26, 1828, "aged 18"; he graduated B.A. 1832, M.A. 1835, and was admitted Foundation Fellow of his College, March 18th, 1834. He died February 16th, 1881, aged (according to the notice in the papers of the time) "73 years."† For twenty-one years he was a mathematical master under the Rev. T. Day, of Brixton, having after a brief trial relinquished the idea of studying for the Bar. He often mentioned that he himself had prepared for the Army fourteen of the officers who fell in the Crimean War.

Mr. Cotterill's connexion with the Society dates from October 16, 1865, when he was elected a member thereof; he became a member of the Council, March 19th, 1866, a position he held until November 14th, 1878, at which date he ceased to attend our meetings.

Mr. Cotterill was an excellent geometer, but his mathematical contributions were not numerous. The causes which operated to bring this result about also account for his (apparently) never having completed to his satisfaction one or two very interesting papers which, we fear, are now irrecoverably lost. The following is as complete a list as we have been able to make of his published and unpublished papers :—

1. A Geometrical Property of Curves of the Third Order; *Cambridge and Dublin Mathematical Journal*, vii., pp. 14—16; 1852.
2. On the Calculus of Distances, Areas, and Volumes, and its relations to the other Forms of Space; *Cambridge and Dublin Mathematical Journal*, vii., pp. 275—279; 1852.
3. Certain Properties of Plane Polygons of an even number of sides; *Proceedings London Mathematical Society*, Vol. i., No. v., pp. 13—16; 1866.
4. Property of a certain Curve described on a Sphere.
5. Some Properties of Cubic Curves.

---

* For other particulars of this gentleman than those given by Mr. Mayor, we are referred by Mr. Leader, of Sheffield, to Gatty's edition of Hunter's "History of Hallamshire" (p. 275)

† Application was made to the present incumbent of Lane End for information as to exact date of birth from Church books, but no answer came to hand.

6. On an Invólution-System of Circular Cubics, and description of the Curve by Points, when the Double Focus is on the Curve; *Proceedings London Mathematical Society*, Vol. i., No. viii. p. 14; 1866.

7. A Goniometrical Problem: to be solved Analytically in one move, or more simply Synthetically in two moves; *Quarterly Journal of Pure and Applied Mathematics*, No. 27; 1865.

8. Property of Six Points on a Plane or Sphere; *Proceedings London Mathematical Society*, Vol. ii., p. 49; 1867.

9. The Eight Points of Intersection of three Quadric Surfaces.

10. Proofs of the Fundamental Formulæ of the Higher Geometry in Space of two and three Dimensions; *Messenger of Mathematics*, Vol. iv., pp. 98—105; 1868.

11. On a Correspondence of Points, such that a curve of the n^{th} order in one plane corresponds to a curve of the $4n^{th}$ in another plane, with three multiple points of the order n on the line of intersection of the planes, and three other multiple points of the order $2n$; *Proceedings London Mathematical Society*, Vol. ii., pp. 119—125; 1868.

12. An Envelope in the Cubic Correspondence of Points.

13. Opposite Points on a Curve.

14. Propositions connected with Residuals.

15. On the Intersection of Plane Curves.

16. On an Algebraical Form, and the Geometry of its Dual connexion with a Polygon, Plane or Spherical; *Proceedings London Mathematical Society*, Vol. iv., pp. 139—143; 1872.

17. On the Correspondence of Points Collinear with a Fixed Origin; *Proceedings London Mathematicdl Society*, Vol. vi., p. 200 (abstract); 1875.

18. On a New View of Pascal's Hexagram; *Proceedings London Mathematical Society*, Vol. viii., p. 311 (abstract); 1877.

Mr. Cotterill was a frequent contributor to the mathematical pages of the *Educational Times*.

R. T.

INDEX.

INDEX OF PAPERS, &c.

[*The mark * indicates that the Paper has either been printed in extenso, or an abstract of it given in the Proceedings or the Appendix.*]

BRIOSCHI, Signor.
*Sur une propriété du Paramètre de la transformée Canonique des Formes cubiques ternaires, pp. 58—63.

CARPMAEL, E., M.A.
*Some Solutions of Kirkman's 15-school-girl Problem, pp. 148—156.

CAYLEY, A., F.R.S.
*The Binomial Equation $x^p - 1 = 0$; Quinquisection, pp. 15, 16.
*On the Flexure and Equilibrium of a Skew Surface, pp. 103—108.
*On the Geodesic Curvature of a Curve on a Surface, pp. 110—117.
*On the Gaussian Theory of Surfaces, pp. 187—192.

COCKLE, Sir JAMES, F.R.S.
*Supplement on Binomial Biordinals, pp. 63—72.

CRAIG, T.
Motion of a Viscous Fluid, p. 14.
*Note on Abel's Theorem, pp. 89—92.

CROFTON, M. W., F.R.S.
*On Operative Symbols in the Differential Calculus, pp. 122—134.

ELLIOTT, E. B., M.A.
*Some Theorems of Kinematics on a Sphere, pp. 47—57.

ELLIS, A. J., F.R.S.
On an apparently Paradoxical Relation of the Circle, Parabola, and Hyperbola, p. 36.

GENESE, R. W., M.A.
*On a System of Coordinates, pp. 157—168.

GLAISHER, J. W. L., F.R.S.
A Partition-Problem connected with a Triangle, and its successive Pedal Triangles, p. 36.
*On some Definite Integrals expressible in terms of the First Complete Elliptic Integral, and of Gamma-functions, pp. 92—99.
Note on the Resolution in Factors of Numbers differing but slightly from π, p. 109.
*Note on certain Symbolic Operators, and their application to the Solution of certain Partial Differential Equations, pp. 201—207.

GROVE, W. B., B.A.
On the Solution of the Inverse Logical Problem, p. 14.

HART, HARRY, M.A.
On the Five Focal Quadrics of a Cyclide, p. 109.
*On the General Equation of the Second Degree in Tetrahedral Coordinates, pp. 135—141.

JEFFERY, H. M., F.R.S.
*On Bicircular Quartics, with a Triple and a Double Focus, and three single Foci, all of them Collinear, pp. 17—26.
*On Spherical Quartics, with a Quadruple Cyclic Arc, and a Triple Focus, pp. 168—177.

McCOLL, H., B.A.
*Note on Prof. C. S. Peirce's Probability Notation of 1867, p. 102.

MANNHEIM, Prof.
Note on the Wave-Surface, p. 156.
*Sur les surfaces parallèles, pp. 177—186.

MERRIFIELD, C. W., F.R.S.
*Considerations respecting the Translation of Series of Observations into Continuous Formulæ, pp. 4—14.
Note on a Kinematical Theorem connected with the Rectilinear Courses of two uniformly sailing Vessels, pp. 36, 213.
*On Ptolemy's Theorem, pp. 142, 214.

MINCHIN, G. M., M.A.
A General Theorem in Kinematics, p. 14.

NIVEN, W. D., M.A.
*On the Electrical Capacity of a Conductor bounded by Two Spherical Surfaces cutting at any angle, pp. 27—36.

ROBERTS, R. A., B.A.
*On the Tangents drawn from a Point to a Nodal Cubic, pp. 99—101.
An Application of Elliptic Functions to the Nodal Cubic, p. 101.
*Note on a System of Cartesian Ovals, passing through Four Points on a Circle, p. 200.

ROBERTS, S., F.R.S.
A Theorem which is the Analogue in Space of the Theorem relating to Three Circles intersecting in a Point, p. 102.
*On Certain Tetrahedra specially related to Four Spheres meeting in a Point, pp. 117—120.
*Historical Note on Dr. Graves's Theorem on Confocal Conics, pp. 120—122.

ROBERTS, W. R. W., M.A.
*On the Periods of the First Class of Hyper-elliptic Integrals, pp. 37—45.
*Note on the Coordinates of a Tangent Line to the Curve of Intersection of two Quadrics, pp. 143—146.

ROUTH, E. J., F.R.S.
*Some Applications of Conjugate Functions, pp. 73—89.

TAYLOR, Rev. C., M.A.
Further remarks on the Geometrical Method of Reversion, pp. 3, 212.

TAYLOR, H. M., M.A.
*The Discrimination of the Maximum or Minimum Path of a Ray of Light reflected at a given Curve, pp. 109, 214.

TEIXEIRA, Prof.
Note sur la Dérivation des Déterminants, pp. 14, 212.

Terry, Rev. T. R., M.A.
A Proof of the Differential Equation which is satisfied by the Hypergeometric Series, p. 36.
The Summation of certain Hypergeometric Series, p. 142.

Walker, J. J., M.A.
*Quaternion Proof of Mr. Samuel Roberts's Theorem of Four Co-intersecting Spheres, pp. 147, 148.
*Theorems in the Calculus of Operations, Part 2, pp. 193—200.

GENERAL INDEX.

Acta Erud. Lips., 121.
Aoust, 117.
Auditor, 1, 14.
Bernoulli, John, 121.
Bertrand, 215.
Bianchi, 62.
Bifolium, 17, 169.
Boole, 67—69, 133.
Bouquet, 215.
Bour, 187, &c.
Brioschi, 215.
Bronwin, 194, 199.
Cambridge Phil. Soc., Trans. of, 215.
Casey, 19, 22, 24, 168, 200.
Cayley, 15, 59, 62, 110, 112, 146, 200.
Chasles, 36, 122, 168, 173, 215.
Clebsch, 168.
Clifford, 215.
Comptes Rendus, 58, 168, 180, 183, 186, 215.
Corrard, 213.
Cotterill, T., 108, 215—217.
Council, Election of, 3.
Crofton, 124, 194, 200, 202, 213.
Darboux, 52, 56, 57, 215.
De Morgan, Memorial Medal Fund, 215.
Diary, Lady's and Gentleman's, 149.
Donkin, 124.
E_a, $E_a - E_b$, Expressions for, 32, 33.
Ecole Polytech., Journal de l', 187.
Educational Times, 200.
Elphinstone, H., 36, 209—212.
Esson, 175.
Euler, 20.
Feuerbach, 160.
Folium, 24, 169.
Freeland, 215.
Frost, A., 215.
Gatty, 217.
Gauss, 95, 110, 187, 188.
Glaisher, J., 10.
Gompertz, 10.
Gournerie, de la, 180.
Graves, 120.
Greatheed, 173.
Greenhill, 84.
Gregory, 206.
Gudermann, 168.
Hargreave, 133, 194, 199.
Harley, 64.
Hart, 139.
Hermite, 58.
Holditch, 47, 57.
Hunter, 217.
Jacobi, 42, 62.
Jeffery, 174, 177, 213.
Jonquières, de, 168.
Journals presented, 3.
——— subscribed for, 3.
Kempe, 47, 52.
Klein, 62, 63.
Legendre, 98.
Leibnitz, 121.
Leudesdorf, 47, 52, 54.
Library, presents made to, 3, 4, 14, 36, 37, 46, 102, 109, 142, 156, 208.
Liouville, 110, 115.
McClintock, 213.
McColl, 102, 215.
MacCullagh, 121.
Mannheim, 181.
Matematica, Annali di, 117, 215.
Mathematical Society, Proc. of, 15, 28, 102, 110, 131, 174, 177, 193, 215.
Mathematical Journal, 73.
Mathematics, American Journal of, 213, 215.
——— Messenger of, 47, 57, 139.
——— Quarterly Journal of, 26, 84, 124, 168, 212, 213, 215.
Mathématique de France, Bulletin de la Société, 177.
Mathématiques de M. Resal, Journal de, 177.
Mathematische Annalen, 24, 62.
Maxwell, 27, 33.

Mayor, J. E. B., 216.
Mécanique Analytique, 103.
Members, Election of, &c., 14, 36, 46, 142, 156.
——— Number of, 1.
Monge, 110.
Muir, 215.
Nature, 215, 216.
Naval Architects, Transactions of Institution of, 8.
Newton, 22, 23, 26, 167.
Niven, W. D., 28.
Non-folium, 169.
Notes and Queries, 216.
Nouvelles Annales, 168, 215.
Obituary, 215.
D'Ocagne, 215.
Papers read, 1—3, 14, 36, 46, 101, 108, 142, 156.
Pappus, 173.
Peaucellier, 215.
Peirce, 102.
Philosophical Magazine, 215.
——— Transactions, 133, 194, 199, 202.
Pirum, 24.
Ramification, Points of, 38.
Region (Gebiet), 90.
Reports, Treasurer's and Secretaries', 1—3.
Reuschle, 15, 16.
Revue Maritime, 213.
Ribaucour, 180, 183, &c.
Riemann, 89.
Roberts, M., 42.
Routh, 75.
Salmon, 59, 99, 101, 110, 121, 179, 188.
Schlömilch, 37, 39.
Sciences Math. et Astr., Bulletin des, 52, 57, 215.
Schwarz, 63.
Scrutators, 3.
Sharp, A., 6.
Stirling, 9.
Sylvester, 200.
Taylor, C., 212.
Terquem, 168.
Thomson, Sir W., 34.
Todhunter, 93.
Unifolium, 17, 169.
Walker, J. J., 131, 193.
Walton, 168.
Warburton, 209.
Wolstenholme, 160,
Woolhouse, 149, &c.
Zeuthen, 24.

Printed by C. F. Hodgson & Son, 1, Gough Square, Fleet Street, London, E.C.

Cr. LONDON MATHEMATICAL SOCIETY.—Financial Report for the Session 1879–80. Dr.

Cr.	£	s.	d.	Dr.	£	s.	d.
Balance brought forward	167	9	0	Postage, Stationery, Printing of Lists, Circulars, and Sundries	21	1	2
2 Subscriptions, 1877–78	2	2	0	Journals	6	18	1
19 do. 1878–79	19	19	0	Rent	10	0	0
35 do. 1879–80	36	14	3	Attendance	6	10	0
1 do. 1880–81	1	1	0	Coals and Gas	1	10	0
13 Entrance Fees	13	13	0	Messrs. Hodgson & Son, account rendered 3 April, 1880, Printing Proceedings, &c....	65	1	6
On account of one Subscription, 1880–1 ...	0	0	9	Messrs. Hodgson & Son, account rendered 3 Nov, 1880, Printing Proceedings, &c. ...	68	7	0
January Dividend on Lord Rayleigh's Fund ...	23	17	0	Balance	194	5	7
January Dividend on Composition Fund ...	8	3	0				
July Dividend on Lord Rayleigh's Fund ...	21	5	11				
July Dividend on Composition Fund	8	12	4				
Sale of Proceedings and Extra Copies... ...	70	16	1				
	£373	13	4		£373	13	4

Three Life Compositions were made in the Session and there were 54 in all; namely:—

28 at £10 each	£280	0	0
26 at £10. 10s.	273	0	0
	£553	0	0

Invested in £586. 14*s*. 4*d*. Consols.

Audited and found correct,

(Signed) R. F. SCOTT.

1880.

ASSETS AND LIABILITIES.

	£	s.	d.		£	s.	d.
Balance in hand	194	5	7	Printing, &c. (say)	36	0	0
5 Subscriptions, 1877–78	5	5	0	Balance	242	5	7
13 do. 1878–79	13	13	0				
62 do. 1879–80	65	2	0				
	£278	5	7		£278	5	7